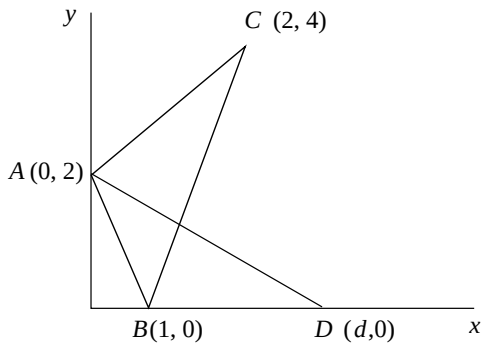
Middle Division: Problems 1

M1.



In the diagram, $D(d, 0)$ lies on the x-axis beyond B. The triangles ABC and ABD have the same area. Determine the value of d.

Solution 1

The triangles both have base AB and the same area. So they must have the same height.

So D must lie on the line through C parallel to AB.

Line AB is $y = -2x + 2$.

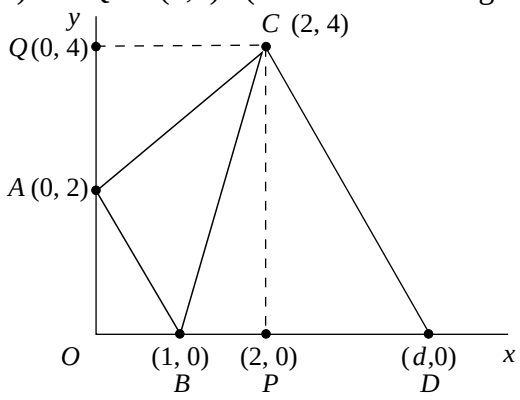
So the parallel line through C is $y = -2x + 8$.

When $y = 0$, $x = 4$.

So $d = 4$.

Solution 2

Let O be the origin (0,0), P be (2, 0) and Q be (0,4). (Shown on the diagram.)



Then

$$\begin{aligned}\text{area } ABC &= \text{area } OPCQ - \text{area } AOB - \text{area } BPC - \text{area } AQC \\ &= 2 \times 4 - \tfrac{1}{2} \times 1 \times 2 - \tfrac{1}{2} \times 1 \times 4 - \tfrac{1}{2} \times 2 \times 2 \\ &= 8 - 1 - 2 - 2 = 3\end{aligned}$$

and

$$\text{area } ABD = \tfrac{1}{2} \times (d - 1) \times 2 = d - 1$$

So $d - 1 = 3$ and therefore $d = 4$.

M2. "Will those in favour of the resolution please hold up their hands?" said the chairperson at a public meeting.

On a count of hands, it appeared that the resolution was carried by a majority of 7. It was then found that, in the excitement of the moment, 6% of those in favour were holding up both their hands. When this had been allowed for, the actual result of the vote proved to be a majority of 2 against the resolution.

How many people were asked to record their vote?

Solution

Let the number of real votes 'for' be x.

Then the number of real votes 'against' is $x + 2$.

With 6% of those voting for holding up two hands, there were apparently $x \times 1.06$ votes 'for'.

The apparent majority was then 7, and so

$$\begin{aligned} x \times 1.06 - (x + 2) &= 7 \\ 0.06x &= 9 \\ x &= 150 \end{aligned}$$

So, in all, there were 302 people voting.

M3. The distance from St Andrews to Leven is 20 miles by one road and 24 miles by another. A cyclist uses one road going out and the other coming back but whichever way she goes her return journey is 2 mph slower than the outward journey. She also found that if she goes out by the longer road and returns on the shorter she takes 6 minutes less time than if she went the other way round.

What is her speed on the outward journey?

Solution

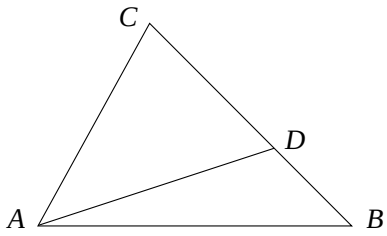
Let outward speed be v mph.

$$\begin{aligned} \frac{20}{v} + \frac{24}{v-2} &= \frac{24}{v} + \frac{20}{v-2} + \frac{1}{10} \\ 200(v-2) + 240v &= 240(v-2) + 200v + v(v-2) \\ v^2 - 2v - 80 &= 0 \\ (v+8)(v-10) &= 0 \\ \text{so} \qquad v &= -8 \text{ or } v = 10 \end{aligned}$$

As her speed must be positive the speed on the outward journey is 10 mph.

M4. In the diagram, $2\angle BAC = 3\angle ABC$ and D lies on BC such that $\angle DAC = 2\angle DAB$. Suppose that $BC = a$, $AC = b$, $AB = c$, $AD = d$ and $CD = e$.

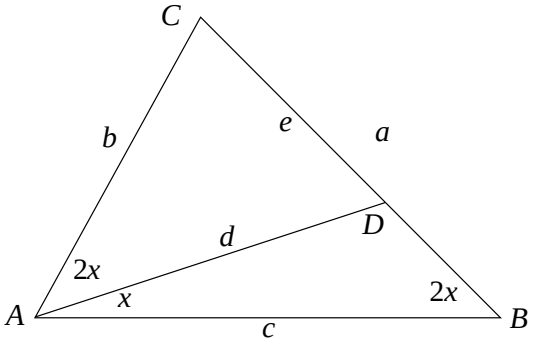
Find expressions for d and e in terms of a, b and c only.



Solution

Let $\angle DAB$ be x.

Then $\angle DAC = 2x$, $\angle BAC = \angle BAD + \angle DAC = 3x$ and $\angle ABC = \frac{2}{3}\angle BAC = 2x$.



Thus triangles ABC and DAC are similar, since $\angle ACB = \angle DCA$ is common, and $\angle ABC = \angle DAC = 2x$.

Hence

$$\frac{AB}{BC} = \frac{DA}{AC} \text{ or } \frac{c}{a} = \frac{d}{b} \qquad (1)$$

$$d = \frac{bc}{a}.$$

Also

$$\frac{CA}{BC} = \frac{CD}{AC} \qquad (2)$$

$$\frac{b}{a} = \frac{e}{b}$$

$$e = \frac{b^2}{a}.$$

The required expressions are $d = \frac{bc}{a}$ and $e = \frac{b^2}{a}$.

M5. A school assembly hall has a rectangular array of chairs. There are exactly 12 boys seated in each of the r rows and exactly 10 girls seated in each of the c columns. There are fewer than 1000 boys and girls in the school. There is just one empty chair.

How many chairs are there in the assembly hall?

Solution

There are $12r$ boys, $10c$ girls and one empty chair.

There are rc chairs in all. So $12r + 10c + 1 = rc \Rightarrow c = \dfrac{12r + 1}{r - 10}$.

Let $r' = r - 10$. Then

$$c = \frac{12r' + 121}{r'} = 12 + \frac{121}{r'}.$$

Since c must be an integer, r' must divide $121 \left(= 11^2\right)$ exactly. Hence

$$r' = 1 \text{ or } 11 \text{ or } 121 \text{ and } r = 11 \text{ or } 21 \text{ or } 131.$$

When $r = 11$, $c = 133$ and when $r = 131$, $c = 13$. In both of these cases there are over 1000 chairs/ children in the school.

So $r = 21$ and $c = 23$, and there are $21 \times 23 = 483$ chairs in the assembly hall.

The empty chair is shown by X, the boys seats by *b* and the unmarked seats are occupied by girls.

	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23
1	X	*b*	*b*	*b*	*b*	*b*	*b*	*b*	*b*	*b*	*b*	*b*	*b*										
2			*b*	*b*	*b*	*b*	*b*	*b*	*b*	*b*	*b*	*b*	*b*	*b*									
3				*b*	*b*	*b*	*b*	*b*	*b*	*b*	*b*	*b*	*b*	*b*	*b*								
4					*b*	*b*	*b*	*b*	*b*	*b*	*b*	*b*	*b*	*b*	*b*	*b*							
5						*b*	*b*	*b*	*b*	*b*	*b*	*b*	*b*	*b*	*b*	*b*	*b*						
6							*b*	*b*	*b*	*b*	*b*	*b*	*b*	*b*	*b*	*b*	*b*	*b*					
7								*b*	*b*	*b*	*b*	*b*	*b*	*b*	*b*	*b*	*b*	*b*	*b*				
8									*b*	*b*	*b*	*b*	*b*	*b*	*b*	*b*	*b*	*b*	*b*	*b*			
9										*b*	*b*	*b*	*b*	*b*	*b*	*b*	*b*	*b*	*b*	*b*	*b*		
10											*b*	*b*	*b*	*b*	*b*	*b*	*b*	*b*	*b*	*b*	*b*	*b*	
11												*b*	*b*	*b*	*b*	*b*	*b*	*b*	*b*	*b*	*b*	*b*	*b*
12	*b*	*b*												*b*	*b*	*b*	*b*	*b*	*b*	*b*	*b*	*b*	*b*
13	*b*	*b*	*b*												*b*	*b*	*b*	*b*	*b*	*b*	*b*	*b*	*b*
14	*b*	*b*	*b*	*b*												*b*	*b*	*b*	*b*	*b*	*b*	*b*	*b*
15	*b*	*b*	*b*	*b*	*b*												*b*	*b*	*b*	*b*	*b*	*b*	*b*
16	*b*	*b*	*b*	*b*	*b*	*b*												*b*	*b*	*b*	*b*	*b*	*b*
17	*b*	*b*	*b*	*b*	*b*	*b*	*b*												*b*	*b*	*b*	*b*	*b*
18	*b*	*b*	*b*	*b*	*b*	*b*	*b*	*b*												*b*	*b*	*b*	*b*
19	*b*	*b*	*b*	*b*	*b*	*b*	*b*	*b*	*b*												*b*	*b*	*b*
20	*b*	*b*	*b*	*b*	*b*	*b*	*b*	*b*	*b*	*b*												*b*	*b*
21	*b*	*b*	*b*	*b*	*b*	*b*	*b*	*b*	*b*	*b*	*b*												*b*