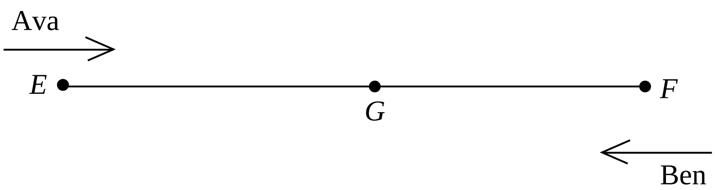
## Senior Division: Problems 2

### S1.

Ava drove from town $E$ to town $F$ at a constant speed of 60 mph. Ben drove from $F$ to $E$ along the same road also at a constant speed. They started their journeys at the same time and passed each other at point $G$.




Ava drove from $G$ to $F$ in 16 minutes. Ben drove from $G$ to $E$ in 25 minutes. Determine Ben's constant speed.

*Solution*

Let Ben's speed be $v$ mph.

A: distance $GF$ = speed × time = $60 \times \dfrac{16}{60} = 16$ miles

B: time $FG = \dfrac{\text{distance}}{\text{speed}} = \dfrac{16}{v}$ hours

A's time from $E$ to $G$ is the same as B's time from $F$ to $G$, so is also $\dfrac{16}{v}$ hours

A: distance $EG$ = speed × time = $60 \times \dfrac{16}{v}$ miles

B: time $GE = \dfrac{\text{distance}}{\text{speed}} = 60 \times \left(\dfrac{16}{v}\right) \div v$ hours

This time is 25 minutes or $\dfrac{25}{60}$ hours.

So

$$60 \times \left(\frac{16}{v}\right) \div v = \frac{25}{60}$$

$$v^2 = 60^2 \times \frac{16}{25}$$

$$v = 60 \times \frac{4}{5} = 48 \text{ since } v \text{ is positive}$$

Ben's constant speed is 48 mph.

## S2.

The numbers $p$, $q$, $r$, $s$ and $t$ are consecutive positive integers arranged in increasing order. $p + q + r + s + t$ is a perfect cube and $q + r + s$ is a perfect square. Find the smallest possible value of $r$.

*Solution*

$$p + q + r + s + t = 5r$$

For this to be a perfect cube, $r = 25k^3$, where $k$ is a positive integer.

$$q + r + s = 3r = 3 \times 25k^3$$

For this to be a perfect square, the smallest value of $k$ is 3. (Next is $k = 12$.)

The smallest $r$ to satisfy both conditions is $r = 25 \times 3^3 = 675$.

*Check*:

$5r = 5 \times 25 \times 3^3 = 15^3$ and $3r = 3 \times 25 \times 3^3 = 45^2$ as required.

## S3.

If $f(x) = x - 3$ and $g\big(f(x)\big) = x^2 - 10$, determine an expression for $g(x)$.

*Solution*

Let $y = f(x) = x - 3$. Then $x = y + 3$ and

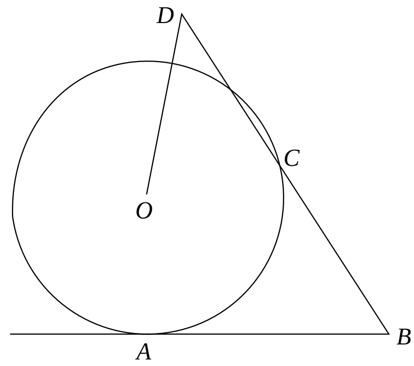
$$\begin{aligned} g\big(f(x)\big) = g(y) &= x^2 - 10 = (y + 3)^2 - 10 \\ &= \big(y^2 + 6y + 9\big) - 10 = y^2 + 6y - 1. \end{aligned}$$

Now replace $y$ by $x$:

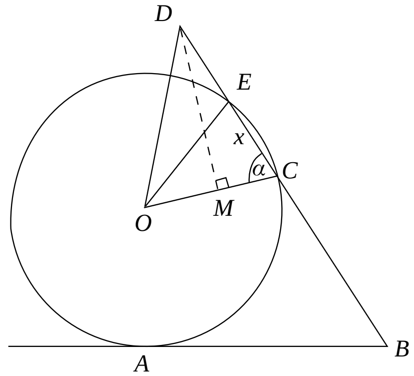
$$g(x) = x^2 + 6x - 1.$$

**S4.**

In the diagram, $AB$ is tangent to the circle with centre $O$ and radius $r$. The length of $AB$ is $p$. Point $C$ is on the circle and $D$ is outside the circle so that $BCD$ is a straight line, as shown. Also $BC = CD = DO = q$. Prove that $p^2 = q^2 + r^2$.




*Solution* 1




Let $CD$ cross the circle at $E$ and let $CE$ be $x$. Let $\angle OCD = \alpha$.
Triangle $ODC$ is isosceles since $OD = DC = q$. Hence the altitude $DM$ will bisect the radius $OC$ ($= r$) at $M$. So $CM = \frac{r}{2}$ and $\cos\alpha = \dfrac{CM}{CD} = \dfrac{\frac{1}{2}r}{q} = \dfrac{r}{2q}$.
Now consider triangle $OEC$, in which $OE = OC = r$.
First we need to find $x$ in terms of $r$ and $q$.
Method 1: using the cosine rule:

$$\begin{aligned} r^2 &= r^2 + x^2 - 2rx\cos\alpha \\ &= r^2 + x^2 - 2rx\frac{r}{2q} \\ &= r^2 + x^2 - \frac{r^2x}{q} \end{aligned}$$

$$x^2 - \frac{r^2x}{q} = 0$$

$$x\left(x - \frac{r^2}{q}\right) = 0$$

Hence either $x = 0$ or $x = \frac{r^2}{q}$.
From the diagram, $x$ is clearly not 0, so $x = \frac{r^2}{q}$.
Method 2: Using the right angled triangle which forms half of the isosceles triangle $OEC$ (much easier if you spot it!):

$$\cos\alpha = \frac{r}{2q} = \frac{\frac{1}{2}x}{r}$$

So $x = \frac{r^2}{q}$.

Now the tangent secant theorem shows that

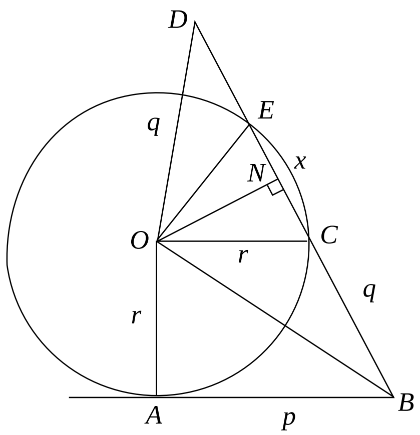
$$BA^2 = BC \times BE \text{ where } BE = q + x = q + \frac{r^2}{q} = \frac{q^2 + r^2}{q}.$$

So

$$p^2 = q \times \frac{q^2 + r^2}{q} = q^2 + r^2$$

as required.

*Solution* 2




Let $CD$ cut the circle again at $E$ and let $CE = x$. Let the base of the altitude of triangle $OCE$ be $N$. Isosceles triangles $DOC$ and $OCE$ are similar as both have the same base angle $DOC = EOC$. So $\frac{\text{side}}{\text{base}} = \frac{q}{r} = \frac{r}{x}$ and hence $x = \frac{r^2}{q}$.

Since triangle $OCE$ is isosceles $N$ is the midpoint of $CE$ and $CN = \tfrac{1}{2}x$. Triangle $ONC$ has a right angle at $N$ and so

$$ON^2 = OC^2 - CN^2 = r^2 - \left(\frac{x}{2}\right)^2.$$

Triangle $ONB$ has a right angle at $N$ and so

$$OB^2 = ON^2 + BN^2 = r^2 - \left(\frac{x}{2}\right)^2 + \left(q + \frac{x}{2}\right)^2 = r^2 + q^2 + qx = 2r^2 + q^2.$$

Triangle $OAB$ has a right angle at $A$ and so

$$OB^2 = OA^2 + AB^2 = r^2 + p^2 = 2r^2 + q^2.$$

Hence $p^2 = r^2 + q^2$ as required.

**S5.**

I have two bags containing coloured balls. The first bag contains five balls, all of which are red, whilst all the balls in the second bag are blue. I transfer one of the balls in the first bag to the second, then pick at random a ball from the second bag and transfer it to the first bag. I now pick a ball at random from the first bag and transfer it to the second bag. If the probability of a ball picked at random from the second bag being blue is then $\frac{3}{5}$, how many blue balls were there in the second bag originally?

*Solution* 1

Suppose there are $n$ blue balls in bag 2 initially. After the first transfer, the contents of the bags are (R = red, B = blue)

bag 1: 4R, 0B and bag 2: 1R, $n$B.

A ball picked at random from bag 2 will be R with probability $1/(n+1)$ and B with probability $n/(n+1)$, giving the contents of the bags after transfer back to bag 1 as

(a) bag 1: 5R, 0B and bag 2: 0R, $n$B with probability $1/(n+1)$

or

(b) bag 1: 4R, 1B and bag 2: 1R, $(n-1)$B with probability $n/(n+1)$.

Assuming (a), transferring a random ball from bag 1 to bag 2 now gives bag 2: 1R, $n$B with probability 1, and then the probability of picking B from bag 2 is $n/(n+1)$. (X)

Assuming (b), transferring a random ball from bag 1 to bag 2 gives

*either*

(b1) bag 2: 2R, $(n-1)$B with probability 4/5,

and then the probability of picking B from bag 2 is $(n-1)/(n+1)$; (Y)

*or*

(b2) bag 2: 1R, $n$B with probability 1/5,

and then the probability of picking B from bag 2 is $n\,(n+1)$. (Z)

The total probability of picking a blue ball from the second bag is thus

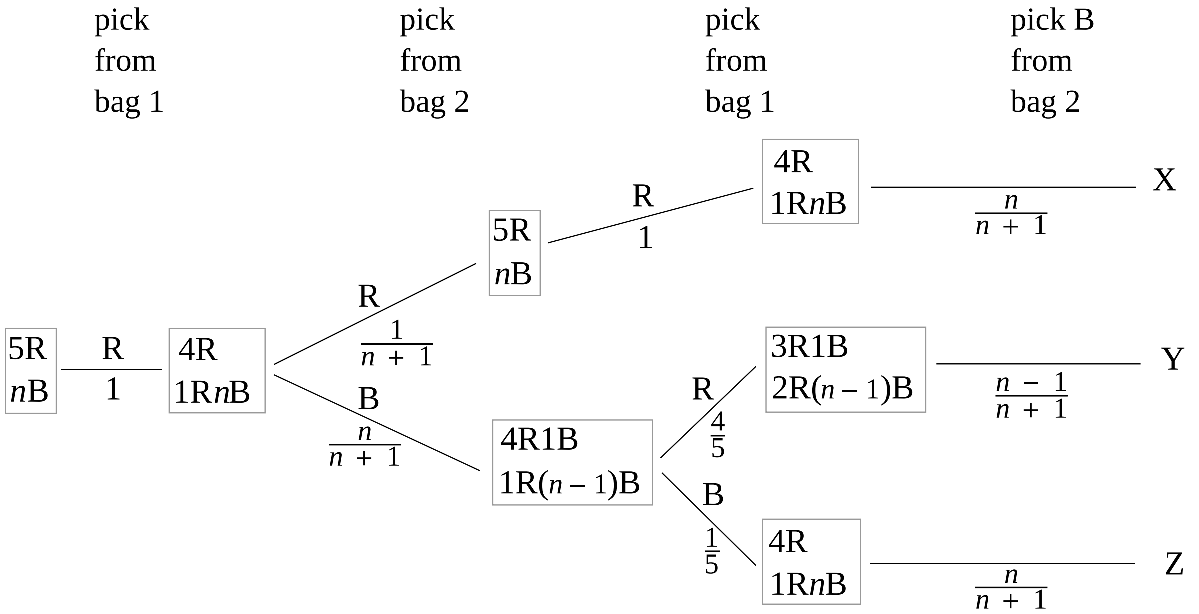
$$\underbrace{\left(\frac{1}{n+1}\times 1\times\frac{n}{n+1}\right)}_{\text{from (X)}}+\underbrace{\left(\frac{n}{n+1}\times\frac{4}{5}\times\frac{n-1}{n+1}\right)}_{\text{from (Y)}}+\underbrace{\left(\frac{n}{n+1}\times\frac{1}{5}\times\frac{n}{n+1}\right)}_{\text{from (Z)}}=\frac{5n^2+n}{5(n+1)^2}.$$

For this to equal 3/5, we need $5n^2 + n = 3\,(n+1)^2$ and hence

$$2n^2 - 5n - 3 = (2n+1)(n-3) = 0,$$

leading to $n = 3$ ($n$ being a positive integer). Thus the second bag originally contained 3 blue balls.

*Solution* 2




On the branches of the tree, R means pick a red ball and B means pick a blue ball. The number below the branch indicates the probability with which that the particular colour is picked. The headers above each level of branching indicate which bag is being picked from, and in the last case that the colour being picked is B i.e. blue.

At the nodes of the tree, in the box, the first line indicates the contents of bag 1 and the second line the contents of bag 2. For example, $\boxed{\begin{array}{l}\text{4R}\\ \text{1R}n\text{B}\end{array}}$ means bag 1 contains 4 red balls and bag 2 contains 1 red ball and $n$ blue balls.

The probability of reaching each of the endpoints X, Y and Z is obtained by multiplying the probabilities of going along all of the branches leading to that endpoint. Hence the probability of reaching X is $1 \times \frac{1}{n+1} \times 1 \times \frac{n}{n+1}$.

The probability of reaching Y is $1 \times \frac{n}{n+1} \times \frac{4}{5} \times \frac{n-1}{n+1} = \frac{4n(n-1)}{5(n+1)^2}$.

The probability of reaching Z is $1 \times \frac{n}{n+1} \times \frac{1}{5} \times \frac{n}{n+1} = \frac{n^2}{5(n+1)^2}$.

Adding these, the total probability of picking a blue ball from bag 2 is $\frac{5n^2 + n}{5(n+1)^2}$.

For this to equal $\frac{3}{5}$, we need $5n^2 + n = 3(n+1)^2$ and hence

$$2n^2 - 5n - 3 = (2n+1)(n-3) = 0,$$

leading to $n = 3$ ($n$ being a positive integer). Thus the second bag originally contained 3 blue balls.