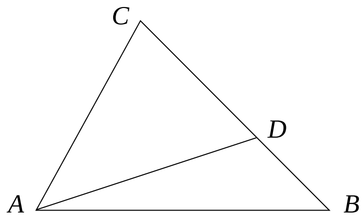
Senior Division: Problems 1

S1.



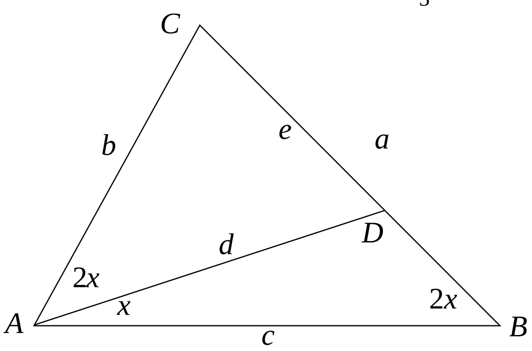
In the diagram, $2\angle BAC = 3\angle ABC$ and D lies on BC such that $\angle DAC = 2\angle DAB$. Suppose that $BC = a$, $AC = b$, $AB = c$, $AD = d$ and $CD = e$.

Find expressions for d and e in terms of a, b and c only.

Solution

Let $\angle DAB$ be x.

Then $\angle DAC = 2x$, $\angle BAC = \angle BAD + \angle DAC = 3x$ and $\angle ABC = \frac{2}{3}\angle BAC = 2x$.



Thus triangles ABC and DAC are similar, since $\angle ACB = \angle DCA$ is common, and $\angle ABC = \angle DAC = 2x$. Hence

$$\frac{AB}{BC} = \frac{DA}{AC} \text{ or } \frac{c}{a} = \frac{d}{b} \tag{1}$$

$$d = \frac{bc}{a}.$$

Also

$$\frac{CA}{BC} = \frac{CD}{AC} \tag{2}$$

$$\frac{b}{a} = \frac{e}{b}$$

$$e = \frac{b^2}{a}.$$

The required expressions are $d = \dfrac{bc}{a}$ and $e = \dfrac{b^2}{a}$.

S2. A school assembly hall has a rectangular array of chairs. There are exactly 12 boys seated in each of the r rows and exactly 10 girls seated in each of the c columns. There are fewer than 1000 boys and girls in the school. There is just one empty chair.

How many chairs are there in the assembly hall?

Solution

There are $12r$ boys, $10c$ girls and one empty chair.

There are rc chairs in all. So $12r + 10c + 1 = rc \Rightarrow c = \dfrac{12r + 1}{r - 10}$.

Let $r' = r - 10$. Then

$$c = \frac{12r' + 121}{r'} = 12 + \frac{121}{r'}.$$

Since c must be an integer, r' must divide $121\ (= 11^2)$ exactly. Hence

$$r' = 1 \text{ or } 11 \text{ or } 121 \text{ and } r = 11 \text{ or } 21 \text{ or } 131.$$

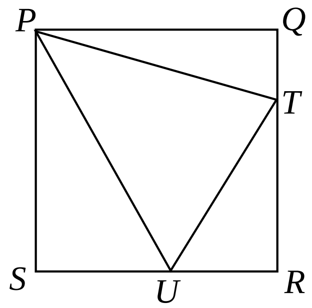
When $r = 11$, $c = 133$ and when $r = 131$, $c = 13$. In both of these cases there are over 1000 chairs/ children in the school.

So $r = 21$ and $c = 23$, and there are $21 \times 23 = 483$ chairs in the assembly hall.

The empty chair is shown by X, the boys seats by *b* and the unmarked seats are occupied by girls.

	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23
1	X	*b*	*b*	*b*	*b*	*b*	*b*	*b*	*b*	*b*	*b*	*b*	*b*										
2			*b*	*b*	*b*	*b*	*b*	*b*	*b*	*b*	*b*	*b*	*b*	*b*									
3				*b*	*b*	*b*	*b*	*b*	*b*	*b*	*b*	*b*	*b*	*b*	*b*								
4					*b*	*b*	*b*	*b*	*b*	*b*	*b*	*b*	*b*	*b*	*b*	*b*							
5						*b*	*b*	*b*	*b*	*b*	*b*	*b*	*b*	*b*	*b*	*b*	*b*						
6							*b*	*b*	*b*	*b*	*b*	*b*	*b*	*b*	*b*	*b*	*b*	*b*					
7								*b*	*b*	*b*	*b*	*b*	*b*	*b*	*b*	*b*	*b*	*b*	*b*				
8									*b*	*b*	*b*	*b*	*b*	*b*	*b*	*b*	*b*	*b*	*b*	*b*			
9										*b*	*b*	*b*	*b*	*b*	*b*	*b*	*b*	*b*	*b*	*b*	*b*		
10											*b*	*b*	*b*	*b*	*b*	*b*	*b*	*b*	*b*	*b*	*b*	*b*	
11												*b*	*b*	*b*	*b*	*b*	*b*	*b*	*b*	*b*	*b*	*b*	*b*
12	*b*	*b*												*b*	*b*	*b*	*b*	*b*	*b*	*b*	*b*	*b*	*b*
13	*b*	*b*	*b*												*b*	*b*	*b*	*b*	*b*	*b*	*b*	*b*	*b*
14	*b*	*b*	*b*	*b*												*b*	*b*	*b*	*b*	*b*	*b*	*b*	*b*
15	*b*	*b*	*b*	*b*	*b*												*b*	*b*	*b*	*b*	*b*	*b*	*b*
16	*b*	*b*	*b*	*b*	*b*	*b*												*b*	*b*	*b*	*b*	*b*	*b*
17	*b*	*b*	*b*	*b*	*b*	*b*	*b*												*b*	*b*	*b*	*b*	*b*
18	*b*	*b*	*b*	*b*	*b*	*b*	*b*	*b*												*b*	*b*	*b*	*b*
19	*b*	*b*	*b*	*b*	*b*	*b*	*b*	*b*	*b*												*b*	*b*	*b*
20	*b*	*b*	*b*	*b*	*b*	*b*	*b*	*b*	*b*	*b*												*b*	*b*
21	*b*	*b*	*b*	*b*	*b*	*b*	*b*	*b*	*b*	*b*	*b*												*b*

S3.

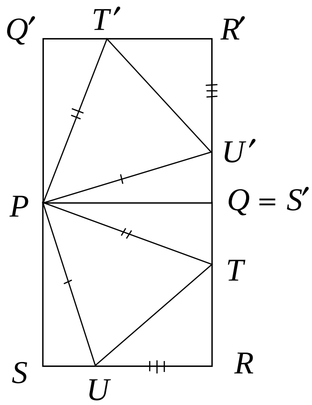


In the diagram, $PQRS$ is a square with sides of length 2. Points T and U are on sides QR and RS respectively such that $\angle TPU = 45°$.

Determine the minimum possible perimeter of triangle RTU.

Solution

Rotate the square $PQRS$ 90° anticlockwise about P, and label corresponding points with a $'$ as shown.



Then $\angle TPU' = \angle UPU' - \angle UPT = 90° - 45° = 45°$.

Also $PU = PU'$.

So triangles TPU and TPU' are congruent (side, angle and common side PT).

Hence $TU = TU'$.

Also UR rotates to $U'R'$ so these lengths are equal.

Thus the perimeter of triangle TUR is equal to the length $RT + TU' + U'R' = RR'$, which is twice the side of the original square i.e. $2 \times 2 = 4$.

The perimeter of triangle RTU is always 4, so its minimum perimeter is also 4.

(If you don't put 'min' in the question then it is easily solved by assuming the length is constant and using a special case.)

S4. George throws three unbiased dice and removes all of the dice that come up with a 5 or 6. Martha then throws the dice that remain, if any. Determine the probability that exactly one of Martha's dice shows a 5 or 6.

Solution

P(5 or 6) = $\frac{2}{6} = \frac{1}{3}$.

P(three dice remain after George's throws) = P(no 5 or 6) = $\left(\frac{2}{3}\right)^3 = \frac{8}{27}$.

P(Martha throws exactly one 5 or 6 with three dice) = $\frac{1}{3} \times \left(\frac{2}{3}\right)^2 \times 3 = \frac{4}{9}$.

P(two dice remain after George's throws) = P(exactly one 5 or 6) = $\frac{1}{3} \times \left(\frac{2}{3}\right)^2 \times 3 = \frac{12}{27}$.

P(Martha throws exactly one 5 or 6 with two dice) = $\frac{1}{3} \times \frac{2}{3} \times 2 = \frac{4}{9}$.

P(one die remains after George's throws) = P(exactly two 5 or 6)

$$= \left(\frac{1}{3}\right)^2 \times \frac{2}{3} \times 3 = \frac{2}{9}.$$

P(Martha throws exactly one 5 or 6 with one die) = $\frac{1}{3}$

If no dice remain after George's throws then Martha cannot throw a 5 or 6.

So the probability that exactly one of Martha's dice shows a 5 or 6 is

$$\frac{8}{27} \times \frac{4}{9} + \frac{12}{27} \times \frac{4}{9} + \frac{2}{9} \times \frac{1}{3}$$

$$= \frac{32 + 48 + 18}{27 \times 9}$$

$$= \frac{98}{243}.$$

S5. The irrational number $\sqrt{2}$ can be written as a series of continued fractions in the following way

$$\sqrt{2} = 1 + \left(\sqrt{2} - 1\right) = 1 + \frac{1}{\sqrt{2} + 1}$$
$$= 1 + \frac{1}{2 + \left(\sqrt{2} - 1\right)} = 1 + \frac{1}{2 + \frac{1}{\sqrt{2} + 1}}$$
$$= 1 + \frac{1}{2 + \frac{1}{2 + \left(\sqrt{2} - 1\right)}} = 1 + \frac{1}{2 + \frac{1}{2 + \frac{1}{\sqrt{2} + 1}}}$$

This process can be continued. If we stop after n steps and ignore the term containing $\sqrt{2}$ we get a rational number $\frac{p_n}{q_n}$. So

$$\frac{p_1}{q_1} = 1, \qquad \frac{p_2}{q_2} = 1 + \frac{1}{2} = \frac{3}{2}, \qquad \frac{p_3}{q_3} = 1 + \frac{1}{2 + \frac{1}{2}} = \frac{7}{5}$$

and so on.

Show that, for all odd integers n, $\frac{p_n}{q_n} < \sqrt{2}$ and for all even n, $\frac{p_n}{q_n} > \sqrt{2}$.

Solution

$\frac{p_n}{q_n} = 1 + \frac{1}{2 + \frac{1}{2 + \frac{1}{2}}}\ldots$ where the denominator is repeated $(n - 1)$ times.

$\frac{p_n}{q_n} + 1 = 2 + \frac{1}{2 + \frac{1}{2 + \frac{1}{2}}}$….. where the right-hand side shows $(n - 1)$ repeats.

$\frac{p_{n+1}}{q_{n+1}} = 1 + \frac{1}{2 + \frac{1}{2 + \frac{1}{2}}}$….. where the right-hand side shows n repeats.

The denominator of the first fraction has only $(n - 1)$ repeats, and so is equal to $\frac{p_n}{q_n} + 1$.

Hence $\frac{p_{n+1}}{q_{n+1}} = 1 + \frac{1}{\frac{p_n}{q_n} + 1}$. (A)

Simplifying $\frac{p_{n+1}}{q_{n+1}} = \frac{p_n + 2q_n}{p_n + q_n}$.

So $\left(\frac{p_{n+1}}{q_{n+1}}\right)^2 - 2 = \left(\frac{p_n + 2q_n}{p_n + q_n}\right)^2 - 2 =$

$$\frac{p_n^2 + 4p_nq_n + 4q_n^2 - 2\left(p_n^2 + 2p_nq_n + q_n^2\right)}{(p_n + q_n)^2} = \frac{q_n^2\left(2 - \left(\frac{p_n}{q_n}\right)^2\right)}{(p_n + q_n)^2}. \qquad \text{(B)}$$

Since the squared terms are always positive, if $\frac{p_n}{q_n} > \sqrt{2}$ then $\frac{p_{n+1}}{q_{n+1}} < \sqrt{2}$ and vice versa.

But we note that $\frac{p_1}{q_1} < \sqrt{2}$ and $\frac{p_2}{q_2} > \sqrt{2}$ so all odd ones are less than $\sqrt{2}$ and all even ones are greater than $\sqrt{2}$.