Senior Division: Problems 1

S1. In the addition sum below, only one out of the five decimal points is in the correct position.

$$\begin{array}{r} 47.5 \\ 38.627 \\ 125.4 \\ 1583.3 \\ \hline 4508.57 \end{array}$$

Find all the possible ways to alter the four incorrect decimal places and make the sum add up correctly.

Solution

Start by assuming the answer of 4508.57 is correct.

To obtain a 7 as the final figure in the answer we must have either

(i) 386.27 or (ii) 12.54 and 158.33.

Case (i) 386.27.

To obtain a 5 as the penultimate figure in the answer we must have 1583.3. Now to get the answer 4508.57, 4750 and 12540 are too much and 475 and 1254, even taken together, are not enough. So no solution is possible.

Case (ii) 12.54 and 158.33.

To obtain a 5 as the penultimate figure in the answer we must have 3862.7. Adding these we get 4033.57, which leaves 475 when subtracted from the total. Here only the answer has the decimal point in its original position, so this is a solution:

$$\begin{array}{r} 475 \\ 3862\cdot 7 \\ 12\cdot 54 \\ 158\cdot 33 \\ \hline 4508\cdot 57 \end{array}$$

Another solution is obtained by moving the decimal point one position to the left (i.e. dividing all numbers by 10) where only 47.5 is unchanged.

A third solution is obtained by moving the decimal point two positions to the left (i.e. dividing all numbers by 100) where only 38.627 is unchanged.

There are no further solutions because either both the remaining two numbers are changed or neither is changed. Hence it is not possible to leave exactly one unchanged.

S2. When Oliver walks briskly down a downward moving escalator he takes 60 steps of the escalator to reach the bottom. When Oliver walks slowly down the escalator at half his previous speed he takes 42 steps of the escalator to reach the bottom. Assuming constant speeds for walking briskly, walking slowly and the movement of the escalator, find how many steps the escalator shows when it is stationary.

Solution

Let the escalator show n steps when it is stationary.

The escalator moves down $n - 60$ steps in the time Oliver walks briskly down 60 steps.

So the escalator moves down 1 step in the time Oliver walks briskly down $\dfrac{60}{n-60}$ steps.

The escalator moves down $n - 42$ steps in the time Oliver walks slowly down 42 steps.

So the escalator moves down 1 step in the time Oliver walks slowly down $\dfrac{42}{n-42}$ steps.

Oliver's slow speed is half his brisk speed so

$$\begin{aligned}
\frac{42}{n-42} &= \frac{1}{2} \times \frac{60}{n-60} \\
42(n-60) &= 30(n-42) \\
42n - 30n &= 60 \times 42 - 30 \times 42 \\
12n &= 30 \times 42 \\
n &= \frac{30 \times 42}{12} = 5 \times 21 = 105
\end{aligned}$$

The escalator shows 105 steps when it is stationary.

Check:

	steps Oliver	steps escalator	as escalator moves 1 step, Oliver moves
brisk	60	$105 - 60 = 45$	$60/45 = 4/3$ steps
slow	42	$105 - 42 = 63$	$42/63 = 2/3$ steps

So Oliver's slow speed is half his brisk speed as required.

S3. Four cards with integers on are placed face down on a table. Five people in succession each take two cards and state the total value of the two cards, but do not reveal the individual values. These totals are

8, 13, 14, 17 and 11.

Determine the numbers on each of the four individual cards.

Solution 12/59

Let the values on the cards be a, b, c and d. If two of these values were the same, say $a = b$, then $a + c = b + c$ and $a + d = b + d$ i.e. two repeated totals. But 5 of the totals are different, and so all the values on the cards are different. Thus we can assume $a < b < c < d$. Also $a + b < a + c < b + c$ and $a + d$. Similarly $d + c > d + b > b + c$ and $a + d$. So the difference between the next to smallest and smallest pair totals is $c - b$, as is the difference between the largest and next to largest.

From 4 cards there are 6 possible pair totals and we have 5 of them. The total of the 6 pair totals is 3 times the total of the individual cards. The total of the 5 pair totals we have is 63, a multiple of 3, and so the missing pair total must also be a multiple of 3.

In ascending order the pair totals are 8 11 13 14 17.

If the missing pair is the two smallest cards, a and b, then $c + d = 17$ and $b + d = 14$, and hence $c = b + 3$. This means that $8 = a + c = a + b + 3$, and so $a + b = 5$. But $a + b$ must be a multiple of 3, which shows that the missing pair is not the two smallest cards.

If the missing pair is the two largest cards, c and d, then $a + b = 8$ and $a + c = 11$, and hence $c = b + 3$. This means that $17 = b + d = c + d - 3$, and so $c + d = 20$. But $c + d$ must be a multiple of 3, which shows that that the missing pair is not the two largest cards.

Thus the missing pair total is a multiple of 3 lying within the sequence of totals: either

8 **9** 11 13 14 17 or

8 11 **12** 13 14 17 or

8 11 13 14 **15** 17

Only the middle sequence has the same difference between the next to smallest and smallest pair totals and between the largest and next to largest. Thus it is the only possible solution.

$c - b = 3$, so $c + b$ must be odd, i.e. $c + b = 13$. Hence $b = 5$, $c = 8$.

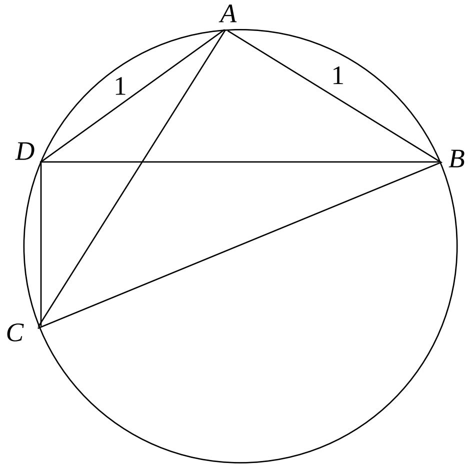
Then $a = 3$, $d = 9$.

The numbers on the cards are 3,5,8 and 9.

S4. Cyclic quadrilateral $ABCD$ has $AB = AD = 1$, $CD = \cos\angle ABC$ and $\cos\angle BAD = -\frac{1}{3}$. Prove that BC is a diameter of the circumscribed circle.

Solution

Use the fact that $\angle BAD$ is obtuse and cos of any angle must be less than 1 to draw a diagram:



Let $\cos\angle ABC = y$.
Then $\cos\angle ADC = \cos\left(180° - \angle ABC\right) = -\cos\angle ABC = -y$.
Use the cosine rule in the two back-to-back triangles ACD and ACB:

$$AC^2 = 1 + y^2 - 2\cdot 1 y\cdot \cos\angle ADC = 1 + y^2 - 2y(-y) = 1 + 3y^2$$

and

$$AC^2 = 1 + BC^2 - 2\cdot 1\cdot BC\cos\angle ABC = 1 + BC^2 - 2BCy.$$

Hence

$$BC^2 - 2BCy - 3y^2 = 0$$
$$\left(BC + y\right)\left(BC - 3y\right) = 0$$
$$BC = -y \text{ or } BC = 3y.$$

But BC must be positive so $BC = 3y$.

Continuation 1:

If BC is a diameter of the circumcircle then triangle BCD must have a right angle at D. Using the cosine rule in triangle BCD

$$\begin{aligned} BD^2 &= y^2 + (3y)^2 - 2\cdot y\cdot 3y\cos\angle BCD \\ &= 10y^2 - 6y^2\cdot\frac{1}{3} \\ &= 8y^2. \end{aligned}$$

Hence

$$BD^2 + CD^2 = 8y^2 + y^2 = 9y^2 = BC^2$$

and triangle BCD satisfies Pythagoras' theorem and *does* have a right angle at D.
Hence BC is a diameter of the circumcircle as required.

Continuation 2:

Use the cosine rule in the two back-to-back triangles BDA and BDC:

$$BD^2 = 1 + 1 - 2\cdot 1\cdot 1\cdot\left(-\frac{1}{3}\right) = \frac{8}{3}$$

and

$$BD^2 = (3y)^2 + y^2 - 2\cdot 3y\cdot y\cdot\frac{1}{3} = 8y^2.$$

Hence

$$y = \frac{1}{\sqrt{3}}.$$

Now triangle ABC has sides $AB = 1$, $BC = 3y = \sqrt{3}$ and $AC^2 = 1 + 3y^2 = 2$.
Hence $AB^2 + AC^2 = 1 + 2 = 3 = BC^2$.
and so by Pythagoras' theorem triangle ABC has a right angle at A.
Hence BC is a diameter of the circumcircle as required.

S5. Choose 6 different non-zero digits.

(a) How many different 6 digit numbers can be formed from these 6 digits?

(b) Find the largest prime factor of the sum of all of these different 6 digit numbers.

Solution

(a) There are 6 ways of choosing the first digit, 5 ways of choosing the second digit, and so on. So there are $6 \times 5 \times 4 \times 3 \times 2 \times 1 = 720$ different possible numbers.

(b) Let the 6 digits be a, b, c, d, e, f.
Each digit will appear in the first position in $\frac{1}{6}$ of the numbers, i.e. $\frac{1}{6} \times 720 = 120$.
Similarly, *each* digit will appear in *each* position 120 times.
An example number is $100000a + 10000b + 1000c + 100d + 10e + f$.
So the total of the numbers will be $111111\,(a + b + c + d + e + f)\,120$.

We need to find the prime factors of each term here:

$$111111 = 11 \times 10101 = 11 \times 3 \times 3367 = 11 \times 3 \times 7 \times 481 = 11 \times 3 \times 7 \times 13 \times 37.$$

The maximum value of $a + b + c + d + e + f$ is $9 + 8 + 7 + 6 + 5 + 4 = 39$. But $39 = 3 \times 13$, so its largest prime factor is less than 37. If the total were $38 = 2 \times 19$, the largest prime factor would be less than 37. If the total were 37 or less, the largest prime factor would not be more than 37. 120 has no prime factor greater than 5.
So the largest prime factor of the sum of all of the 6 digit numbers is 37.