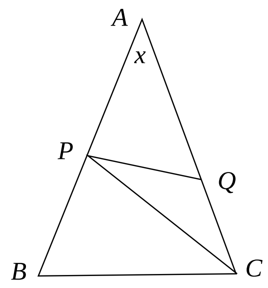
Middle Division: Problems 2

M1.

Triangle ABC has $AB = AC$ and $\angle BAC = x$ is less than $60°$. Point P lies on AB such that $CB = CP$. Point Q lies on AC such that $CQ = PQ$.

Determine $\angle CQP$ in terms of x.



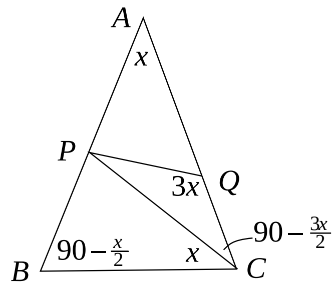
Solution

$\angle ABC = \angle ACB = \frac{1}{2}(180 - x) = 90 - \frac{1}{2}x$
(base angles of isosceles triangle ABC equal)

Triangles ABC and CBP are similar (both isosceles with same base angle at B)

So $\angle BCP = \angle BAC = x$.

$\angle QCP = \angle ACB - \angle BCP = \left(90 - \frac{1}{2}x\right) - x = 90 - \frac{3}{2}x$.



Triangle CQP is isosceles, so its apex angle $\angle CQP$ is

$$180 - 2 \times \text{base angle} = 180 - 2\left(90 - \tfrac{3}{2}x\right) = 3x.$$

M2.

One hundred years ago there was a gathering to present an award to a local teacher in recognition of many years of service.

The women there numbered four-fifths of the men, 40% of whom were unmarried. Half of the married women were accompanied by their husbands and a quarter of the married men by their wives. Thirty of the women were unmarried.

How many people were there at the gathering?

Solution

If the number of men is m then

	Men (m)	Women
Married	$\frac{3}{5}m$	$\frac{4}{5}m - 30$
Single	$\frac{2}{5}m$	30
Total	m	$\frac{4}{5}m$

However, half the married women were with their husbands which must be the same number as quarter of the married men who came with their wives.

$$\frac{2}{5}m - 15 = \frac{3}{20}m$$

$$\frac{1}{4}m = 15$$

$$m = 60$$

There were 60 men and 48 women. There were 108 people at the gathering.

M3.

A market trader sells fruit. As a special offer, he has made up baskets of fruit. The first has four bananas, three oranges and two apples and costs £2.90; the second has three bananas, two oranges and four apples and costs £2.60; the third has two bananas, four oranges and three apples and costs £2.60. Individual fruit bought costs 20% more than the corresponding price in any of the special offer baskets. I don't fancy any of the mixtures in the baskets, but I do want at least two bananas, two oranges and three apples. So I buy bananas, oranges and apples individually and pay £3.12. How many of each fruit do I get?

Solution

Let B, O and A denote the price in pence of bananas, oranges and apples respectively in the special offer baskets. So we have

$$4B + 3O + 2A = 290$$

$$3B + 2O + 4A = 260$$

$$2B + 4O + 3A = 260.$$

Solving these we get $B = 40$, $O = 30$ and $A = 20$.

So if b, o and a denote the prices of the fruit bought individually, then

$$b = \frac{6}{5} \times B = 48, \qquad o = \frac{6}{5} \times 30 = 36, \qquad a = \frac{6}{5} \times 20 = 24.$$

Suppose I buy x bananas, y oranges and z apples then

$$48x + 36y + 24z = 312.$$

Dividing by 12 gives

$$4x + 3y + 2z = 26.$$

We have $x \geqslant 2$, $y \geqslant 2$ and $z \geqslant 3$.

$$4(x - 2) + 3(y - 2) + 2(z - 3) = 6.$$

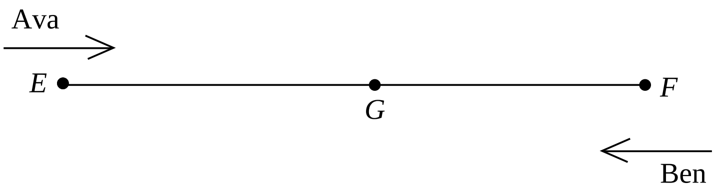
There are three possibilities for this:

(a) $x = 3, y = 2, z = 4$ (b) $x = 2, y = 4, z = 3$ (c) $x = 2, y = 2, z = 6$.

But I wanted a mixture different from what the trader was offering in the special offers and only (c) is different. So I buy 2 bananas, 2 oranges and 6 apples.

M4.

Ava drove from town E to town F at a constant speed of 60 mph. Ben drove from F to E along the same road also at a constant speed. They started their journeys at the same time and passed each other at point G.



Ava drove from G to F in 16 minutes. Ben drove from G to E in 25 minutes. Determine Ben's constant speed.

Solution

Let Ben's speed be v mph.

A: distance GF = speed × time = $60 \times \dfrac{16}{60} = 16$ miles

B: time $FG = \dfrac{\text{distance}}{\text{speed}} = \dfrac{16}{v}$ hours

A's time from E to G is the same as B's time from F to G, so is also $\dfrac{16}{v}$ hours

A: distance EG = speed × time = $60 \times \dfrac{16}{v}$ miles

B: time $GE = \dfrac{\text{distance}}{\text{speed}} = 60 \times \left(\dfrac{16}{v}\right) \div v$ hours

This time is 25 minutes or $\dfrac{25}{60}$ hours.

So

$$\begin{aligned} 60 \times \left(\frac{16}{v}\right) \div v &= \frac{25}{60} \\ v^2 &= 60^2 \times \frac{16}{25} \\ v &= 60 \times \frac{4}{5} = 48 \text{ since } v \text{ is positive} \end{aligned}$$

Ben's constant speed is 48 mph.

M5.

The numbers p, q, r, s and t are consecutive positive integers arranged in increasing order. $p + q + r + s + t$ is a perfect cube and $q + r + s$ is a perfect square. Find the smallest possible value of r.

Solution

$$p + q + r + s + t = 5r$$

For this to be a perfect cube, $r = 25k^3$, where k is a positive integer.

$$q + r + s = 3r = 3 \times 25k^3$$

For this to be a perfect square, the smallest value of k is 3. (Next is $k = 12$.)

The smallest r to satisfy both conditions is $r = 25 \times 3^3 = 675$.

Check:

$5r = 5 \times 25 \times 3^3 = 15^3$ and $3r = 3 \times 25 \times 3^3 = 45^2$ as required.