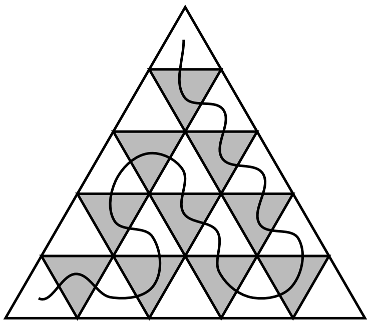
Senior Division: Problems 2

S1. A large equilateral triangle with sides of integer length N is split into small equilateral triangular cells each with side length 1 by drawing lines parallel to its sides. A continuous track starts in the cell at one corner of the large triangle and moves from cell to cell, always crossing at an edge shared by the two cells. The track never revisits a cell. Find, with proof, the greatest number of cells that can be visited on one track.

Solution

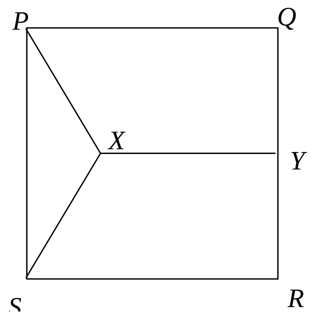


Colour the cells alternately grey and white as shown when $N = 5$. For a large triangle with side N there are always T_N white cells and T_{N-1} grey cells, where T_N is the N th triangular number. $T_N - T_{N-1} = N$ so there are N more white cells than grey cells.

Every possible track alternates between white and grey cells, always starting on a white cell. So once all the grey cells have been visited, with a track starting and ending on a white cell, there must be $N - 1$ white cells remaining. A track of this length is always possible, by missing the final white cell at the end of each row, apart from the last cell (i.e. missing $N - 1$ white cells in all), as shown in the diagram.

The longest track passes through T_{N-1} grey cells and $T_{N-1} + 1$ white cells, where $T_{N-1} = \frac{N}{2}(N - 1)$. Hence the maximum track length is $N(N - 1) + 1$.

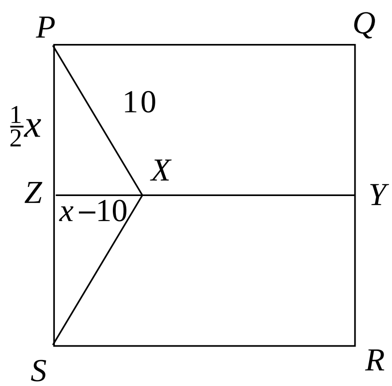
S2. In the diagram, $PQRS$ is a square with XY perpendicular to QR and $XP = XS = XY = 10$ cm. What is the area of the square?



Solution

Let the side of the square be x cm.

Construct XZ perpendicular to PS.



Apply Pythagoras' theorem to the right-angled triangle PXZ

$$ZX^2 + ZP^2 = PX^2$$

$$(x - 10)^2 + \left(\frac{x}{2}\right)^2 = 10^2$$

$$x^2 - 20x + 100 + \frac{x^2}{4} = 100$$

$$\frac{5x^2}{4} - 20x = 0$$

$$x(5x - 80) = 0$$

So either $x = 0$ (the triangle collapses, which it clearly does not) or $x = 16$.

So the area of the square is $16^2 = 256$ cm^2.

S3. If x is a real number satisfying $x^3 + \frac{1}{x^3} = 2\sqrt{5}$, determine the exact value of $x^2 + \frac{1}{x^2}$.

Hint: Start by looking at $\left(x + \frac{1}{x}\right)^2$.

Solution

$$\left(x + \frac{1}{x}\right)^2 = x^2 + 2 + \frac{1}{x^2} \tag{1}$$

$$\left(x + \frac{1}{x}\right)\left(x^2 + \frac{1}{x^2}\right) = x^3 + \frac{1}{x^3} + x + \frac{1}{x} \tag{2}$$

Let $y = x + \frac{1}{x}$ and $z = x^2 + \frac{1}{x^2}$. Then

$$y^2 = z + 2$$

$$yz = 2\sqrt{5} + y$$

so that

$$y\left(y^2 - 2\right) = 2\sqrt{5} + y$$

$$y^3 - 3y - 2\sqrt{5} = 0 \tag{3}$$

But we can see that $y = \sqrt{5}$ is a solution so $\left(y - \sqrt{5}\right)$ is a factor and the left-hand side can be factorised to give

$$\left(y - \sqrt{5}\right)\left(y^2 + \sqrt{5}y + 2\right) = 0.$$

We now try to solve $y^2 + \sqrt{5}y + 2 = 0$:

$$y = \frac{-\sqrt{5} \pm \sqrt{5 - 4 \times 2}}{2} = \frac{-\sqrt{5} \pm \sqrt{-3}}{2}$$

so there are no more real solutions.

Finally $z = y^2 - 2 = \left(\sqrt{5}\right)^2 - 2 = 5 - 2 = 3$.

So the exact value of $x^2 + \frac{1}{x^2}$ is 3.

S4. x and y are positive integers such that $x^2 + y^2 - x$ is exactly divisible by $2xy$.

(a) Find all possible values for y when $x = 9$.

(b) Show that x must always be a perfect square.

Solution

Note: this solution uses the symbol | to mean ‘divides into with integer result’.

(a)

When $x = 9$, $x^2 + y^2 - x = 72 + y^2$ so $18y \mid (72 + y^2)$.

Since $18 \mid 72$, $18 \mid y^2$ and hence $6 \mid y$ i.e. $y = 6m$ for some positive integer m. So

$$18(6m) \mid (72 + 36m^2)$$
$$3m \mid (2 + m^2).$$

As $m \mid m^2$, m must divide 2.

When $m = 1$, $3 \mid 3$ so $y = 6$ is a solution.

When $m = 2$, $6 \mid 6$ so $y = 12$ is another solution.

And there are no other possible values for y.

(b) $x^2 + y^2 - x$ must be exactly divisible by each factor separately.

$x \mid (x^2 + y^2 - x)$ so $x \mid y^2$ so $y^2 = kx$ for some positive integer k.

If all the prime factors of x are raised to even powers then x must be a perfect square.

If any prime $p \mid x$ then $y^2 = kx$ so $p \mid y^2$ and hence $p \mid y$.

Thus $2xy = np^2$ for some positive integer n so $p^2 \mid (x^2 + y^2 - x)$ and $p^2 \mid x$.

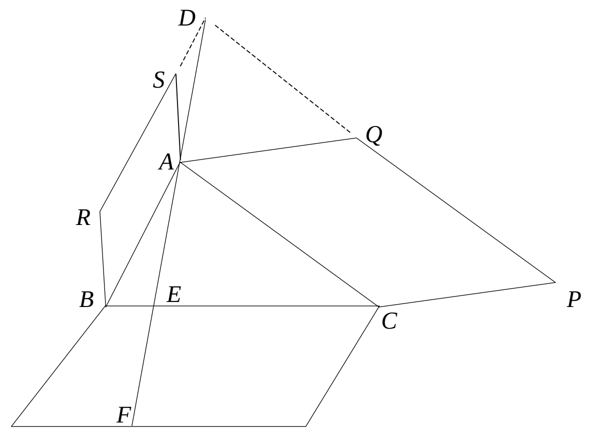
If $p^3 \mid x$ then $y^2 = kx$ so $p^3 \mid y^2$ and hence $p^2 \mid y$.
Thus $2xy = np^5$ for some positive integer n so $p^5 \mid (x^2 + y^2 - x)$. p^4 divides both x^2 and y^2 and so $p^4 \mid x$.

If for some positive integer i, $p^{2i+1} \mid x$ then $y^2 = kx$ so $p^{2i+1} \mid y^2$ and hence $p^{i+1} \mid y$.
Thus $2xy = np^{3i+2}$ for some positive integer n so $p^{3i+2} \mid (x^2 + y^2 - x)$. p^{2i+2} divides both x^2 and y^2 and so $p^{2i+2} \mid x$.

Thus if a prime to an odd power divides x so does the prime to the next higher even power.
Thus x is a product of primes to even powers, i.e. x is a perfect square.

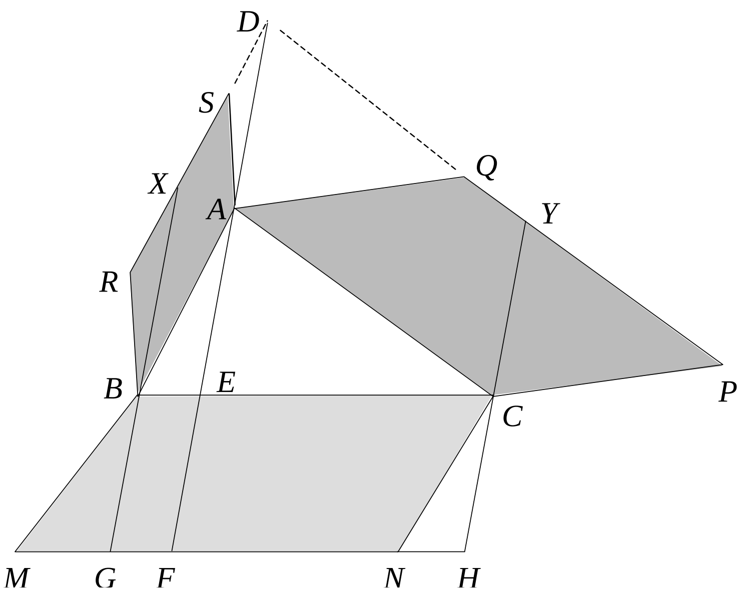
S5. Triangle *BAC* has a right angle at *A*. Any two parallelograms, *ACPQ* and *ABRS* are constructed on *AC* and *AB* respectively. The lines *PQ* and *RS* are produced to meet at *D*. The line *DAEF* is drawn with *DA* = *EF*.

Show that the area of any parallelogram with side *BC* and *F* lying on the opposite side equals the sum of the areas of the parallelograms *ABRS* and *ACPQ*.



Solution

Draw lines through *B* and *C*, parallel to *DF*, as shown.



Parallelograms with the same (or equal) bases and between the same pair of parallel lines have the same area.

The parallelograms *ADYC* and *AQPC* have equal areas, as do parallelograms *BADX* and *BASR*.

Since *DA* = *EF*, parallelograms *ADYC* and *EFHC* have equal areas and *DABX* and *EFGB* also have equal areas.

Triangles *BGM* and *CHN* are congruent hence

Area of parallelogram *BMNC* = Area of *ABRS* + Area of *ACPQ*