2020-2021 Senior Division: Problems 1: Solutions

S1. A parent has washed some nappies in a strong bleach solution and wishes to rinse them so that they contain as weak a bleach solution as possible. By wringing out, the nappies can be made to contain just half a litre of solution. Show that two thorough rinses, such that the solution strength is uniform, the first using 12 litres of water and the second using 8 litres of water, reduces the strength of the bleach solution to $\frac{1}{425}$ of its original value.

If 20 litres of clean water is all that is available and the parent is prepared to do only two rinses, how best should the water be divided between the two rinses?

Solution

In rinse 1, 12.5 litres of water with bleach is reduced to 0.5 litres, i.e. to $\frac{0.5}{12.5} = \frac{1}{25}$ of its original value.

In rinse 2, 8.5 litres of water with bleach is reduced to 0.5 litres, i.e. to $\frac{0.5}{8.5} = \frac{1}{17}$ of its original value.

So the two rinses in succession reduce to $\frac{1}{25} \times \frac{1}{17} = \frac{1}{425}$ of its original value.

Use x litres of water for the first rinse. In rinse 1 $(x + \frac{1}{2})$ litres of water with bleach is reduced to $\frac{1}{2}$ litre, i.e. to $\frac{\frac{1}{2}}{x + \frac{1}{2}} = \frac{1}{2x + 1}$ of its original value.

In rinse 2 $(20 - x + \frac{1}{2})$ litres of water with bleach is reduced to $\frac{1}{2}$ litre, i.e. to $\frac{\frac{1}{2}}{20 - x + \frac{1}{2}} = \frac{1}{41 - 2x}$ of its original value.

So the two rinses in succession reduce to $\frac{1}{(2x + 1)(41 - 2x)}$ of its original value.

Minimise this ratio by maximising

$$\begin{aligned}(2x + 1)(41 - 2x) &= -4x^2 + 80x + 41 \\ &= -4(x - 10)^2 + 441.\end{aligned}$$

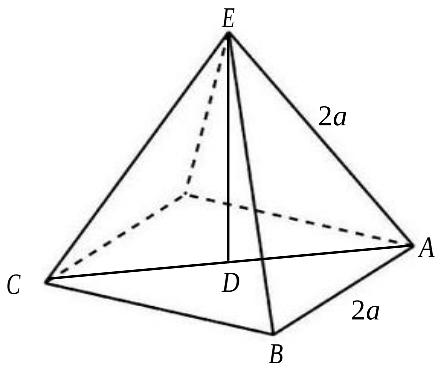
This is maximum when $x - 10 = 0$ i.e. $x = 10$.

So it is best to divide the water equally between the two rinses, which will reduce the bleach concentration to $\frac{1}{441}$ of its original value.

S2. A pyramid has a square base and four equilateral triangles as its other faces. The four equilateral triangles can also make a tetrahedron. What is the ratio of the volumes of the pyramid and the tetrahedron? **Justify your answer.**

Solution

Let the side length of the square be $2a$.
Therefore, the side length of the triangles is also $2a$.
Volume of a pyramid = $\frac{1}{3}$ × base area × height.



Area of the base = $4a^2$.
Consider the triangle ABC.

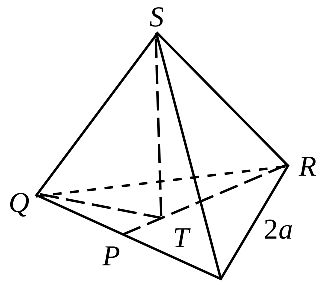
$$\begin{aligned} AC^2 &= (2a)^2 + (2a)^2 \\ &= 4a^2 + 4a^2 \\ &= 8a^2 \\ AC &= \sqrt{8a^2} = 2\sqrt{2}\,a \end{aligned}$$

Now consider triangle CDE, where $CD = \frac{1}{2}AC$.

$$\begin{aligned} DE^2 &= CE^2 - CD^2 \\ &= 4a^2 - 2a^2 \\ &= 2a^2 \\ DE &= \sqrt{2}\,a. \end{aligned}$$

Volume of the pyramid $= \frac{1}{3} \times$ base area $\times$ height $= \frac{1}{3} \times 4a^2 \times \sqrt{2}a = \frac{4\sqrt{2}}{3}a^3$.

Now consider the regular tetrahedron and triangle PQR.



$$PR^2 = QR^2 - PQ^2 = 4a^2 - a^2 = 3a^2$$

$$PR = \sqrt{3}a$$

Area of pyramid base = $\frac{1}{2} \times 2a \times \sqrt{3}a = \sqrt{3}a^2$

Now consider triangle PQT where $PT = \frac{1}{3}PR = \frac{1}{3}\sqrt{3}a$

$$\begin{aligned} QT^2 &= PQ^2 + PT^2 \\ &= a^2 + \frac{1}{3}a^2 = \frac{4}{3}a^2 \\ QT &= \frac{2\sqrt{3}}{3}a \end{aligned}$$

Consider triangle QST:

$$ST^2 = QS^2 - QT^2 = 4a^2 - \frac{4}{3}a^2 = \frac{8}{3}a^2$$

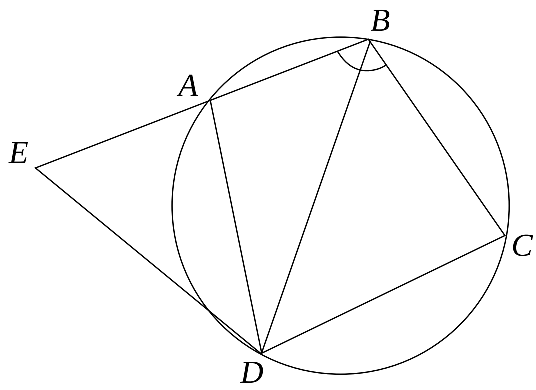
$$ST = \frac{2\sqrt{2}\sqrt{3}}{3}a.$$

Volume of the tetrahedron = $\frac{1}{3} \times \sqrt{3}a^2 \times \frac{2\sqrt{2}\sqrt{3}}{3}a = \frac{2\sqrt{2}}{3}a^3$.

$$\frac{\text{Volume of the pyramid}}{\text{Volume of the tetrahedron}} = \frac{\frac{4\sqrt{2}}{3}a^3}{\frac{2\sqrt{2}}{3}a^3} = 2.$$

S3. $ABCD$ is a cyclic quadrilateral. BD bisects $\angle ABC$. Extend the side BA beyond A to a point E. Show that $DE = DB$ if and only if $AE = BC$.

Solution



$AD = CD$ (subtend same angle)
$\angle BCD = 180° - \angle BAD$ (opposite angles of cyclic quadrilateral supplementary)
$\angle DAE = 180° - \angle BAD = \angle BCD$

If $DE = DB$, then triangle EDB is isosceles and so $\angle DEB = \angle DBE = \angle DBC$.
Then triangles DEA and DBC are congruent (AAS).
Hence $EA = BC$ as required.

If $AE = BC$, then triangles AED and CBD are congruent (SAS) and so $DE = DB$ as required.

S4. A coin is biassed so that the probability of obtaining a head is p, where $0 < p < 1$. If the sequence HHH occurs first then Player A wins, and if the sequence HTH occurs first then player B wins. The coin is tossed until one player wins. For what value of p is the game fair?

(A fair game is one in which both players have an equal chance of winning.)

Solution

Note that if two tails occur consecutively the game is reset as if no tosses had occurred. The possible outcomes are

		probability
HHH	A wins	p^3
HHTH	B wins	$p^3(1 - p)$
HHTT	reset	
HTH	B wins	$p^2(1 - p)$
HTT	reset	
THHH	A wins	$p^3(1 - p)$
THHTH	B wins	$p^3(1 - p)^2$
THHTT	reset	
THTH	B wins	$p^2(1 - p)^2$
THTT	reset	
TT	reset	

The total probability that A wins must equal the total probability that B wins, so

$$p^3 + p^3(1 - p) = p^3(1 - p) + p^2(1 - p) + p^3(1 - p)^2 + p^2(1 - p)^2$$

$$p^2\left(p^3 - p^2 - 3p + 2\right) = 0$$

p is positive so

$$(p - 2)\left(p^2 + p - 1\right) = 0$$

p must be less than 1, so

$$p^2 + p - 1 = 0$$

$$p = \tfrac{1}{2}\left(-1 \pm \sqrt{(1 + 4)}\right)$$

p must be positive, so

$$p = \tfrac{1}{2}\left(\sqrt{5} - 1\right)$$

Check: With this value of p, P(A wins) = P(B wins) = 0.326 approx. and P(reset) = 0.3475 approx. and the sum of these probabilities is 1.

S5. Given that $f(x) = x^2 - x - 1$, $g(x) = ax + b$ and $f\left(g(x)\right) = 4x^2 - 10x + 5$, determine all possible pairs of values of a and b which satisfy this relationship.

Solution

$$\begin{aligned} f(g(x)) &= (ax + b)^2 - (ax + b) - 1 \\ &= a^2x^2 + 2abx + b^2 - ax - b - 1 \\ &= a^2x^2 + a(2b - 1)x + b^2 - b - 1. \end{aligned}$$

Equating coefficients:

x^2: $a^2 = 4$ so $a = 2$ or $a = -2$

x: $a(2b - 1) = -10$

1: $b^2 - b - 1 = 5$

$b^2 - b - 6 = 0$

$(b - 3)(b + 2) = 0$

$b = 3$ or $b = -2$

When $b = 3$, $a(2b - 1) = 5a = -10$ so $a = -2$

When $b = -2$, $a(2b - 1) = -5a = -10$ so $a = 2$

So there are two solution pairs: $a = -2$ with $b = 3$ and $a = 2$ with $b = -2$.

Alternative to equating coefficients:

Since the expressions are valid for all x, we can choose particular values:

Let $x = 0$, then $f\left(g(0)\right) = b^2 - b - 1 = 5 \Rightarrow b = 3$ or $b = -2$.

Let $x = 1$, then $f(g(1)) = a^2 + a(2b - 1) + 5 = 4 - 10 + 5 = -1$

$$a^2 + a(2b - 1) + 6 = 0 \qquad \text{(i)}$$

Let $x = -1$. Then $f\left(g(-1)\right) = a^2 - a(2b - 1) + 5 = 4 + 10 + 5$

$$a^2 - a(2b - 1) - 14 = 0 \qquad \text{(ii)}$$

(i) − (ii):

$$2a(2b - 1) + 20 = 0$$

$$\text{When } b = 3,\ 2a(5) + 20 = 0 \Rightarrow a = -2$$

$$\text{When } b = -2,\ 2a(-5) + 20 = 0 \Rightarrow a = 2$$

So there are two solution pairs, as above.