Senior Division: Solutions to Problems 2

S1. A positive integer ends in the digit 4 and has the property that it becomes four times as large when the 4 is moved from the end and placed at the front. What is the smallest such number?

Solution

Let the number be N and have $n + 1$ digits. So $N = 10x + 4$ where x has n digits.
Moving the 4 from the end to the front gives the number $4 \times 10^n + x$. So we have the equation

$$4(10x + 4) = 4 \times 10^n + x.$$

Rearranging gives that

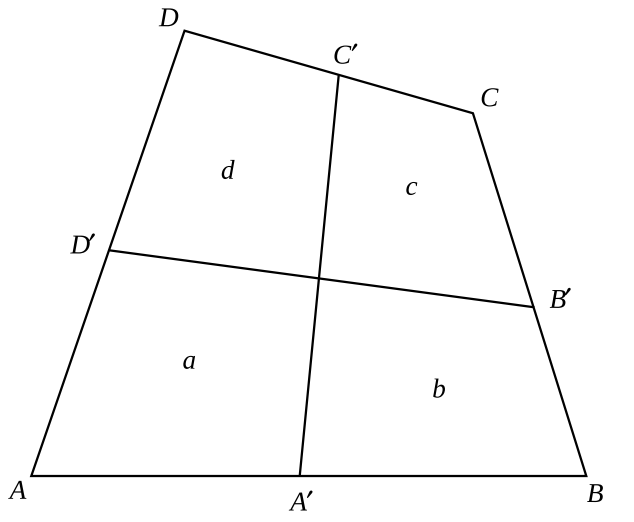
$$39x = 4\left(10^n - 4\right).$$

Thus 39 must divide $10^n - 4$.

Value of n	1	2	3	4	5
Value of $10^n - 4$	6	96	996	9996	99996

The smallest n for which this holds is $n = 5$ in which case $4\left(10^5 - 4\right) = 39 \times 10256$.
Check that $4 \times 102564 = 410256$. So the smallest such number is 102564.

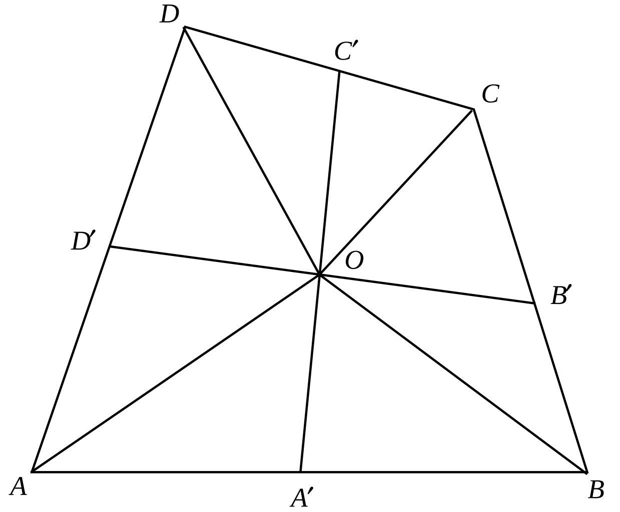
S2.



Let $ABCD$ be a quadrilateral. Let A' be the midpoint of AB, B' the mid-point of BC, C' the mid-point of CD and D' the mid-point of AD. Draw the lines $A'C'$ and $B'D'$ and let a, b, c, d be the areas of the four minor quadrilaterals as shown in the figure. Prove that $a + c = b + d$.

Solution

From the point of intersection O of the lines $A'C'$ and $B'D'$, draw the lines to each of the vertices of the quadrilateral. Recall that the area of a triangle is ½ × *base* × *height.*



Now triangles OAA' and OBA' have equal bases and the same height and so have the same area. The same applies to the pairs of triangles (OBB', OCB'), (OCC', ODC') and (ODD', OAD').
So $a + c$ = area of triangle OAA' + area of triangle OAD' + area of triangle OCB' + area of triangle OCC' = area of triangle OBA' + area of triangle ODD' + area of triangle OBB' + area of triangle $ODC' = b + d$.

S3. Find all values of x such that

$$\log_2(3x + 2) + \log_2(4x - 4) = 3.$$

Solution

$$\begin{aligned} \log_2\big((3x + 2)(4x - 4)\big) &= 3 \\ (3x + 2)(4x - 4) &= 2^3 = 8 \\ (3x + 2)(x - 1) &= 2 \\ 3x^2 - x - 2 &= 2 \\ 3x^2 - x - 4 &= 0 \\ (3x - 4)(x + 1) &= 0 \\ x &= \frac{4}{3} \text{ or } -1 \end{aligned}$$

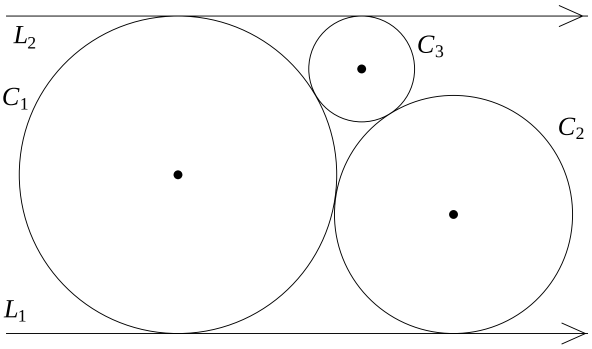
When $x = \frac{4}{3}$, $3x + 2 = 6$ and $4x - 4 = \frac{4}{3}$. Then $\log_2 6 + \log_2 \frac{4}{3} = \log_2\left(6 \times \frac{4}{3}\right) = \log_2 8 = 3$ so this is a solution of the original equation.

When $x = -1$, $3x + 2 = -1$ and $4x - 4 = -8$ so this is not a solution of the original equation, because we cannot take logs of negative numbers.

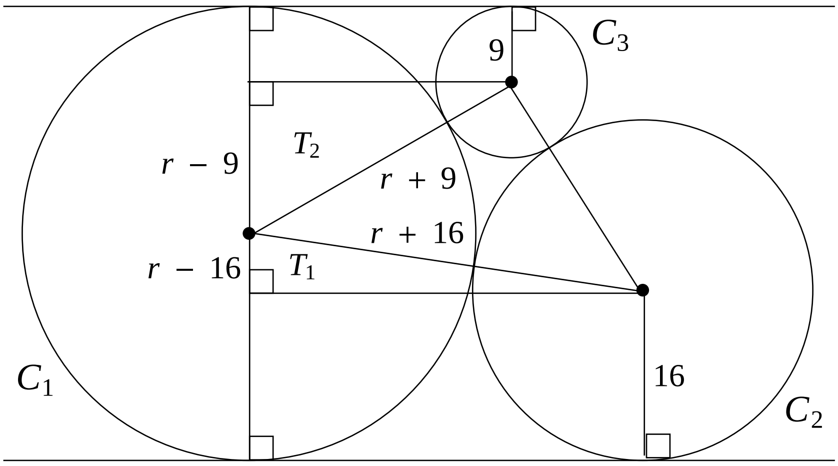
So $x = \frac{4}{3}$ is the only solution of the original equation.

S4.

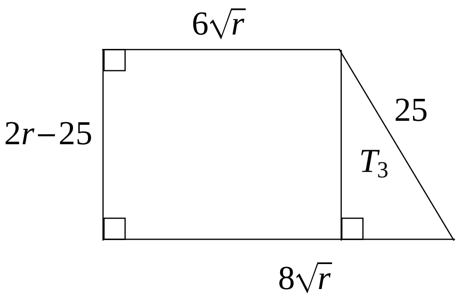
The diagram shows three circles, C_1, C_2 and C_3 and two parallel lines, L_1 and L_2. C_1 touches both the lines, C_2 touches L_1 and C_1, and C_3 touches C_1, C_2 and L_2. The radius of C_2 is 16 and the radius of C_3 is 9. Find the radius of C_1.



Solution



Let the radius of C_1 be r.
In triangle T_1, two sides have lengths $r + 16$ and $r - 16$. The remaining side has length a where $a^2 = (r + 16)^2 - (r - 16)^2 = 2 \times 32r = 64r$. So $a = 8\sqrt{r}$.
In triangle T_2, two sides have lengths $r + 9$ and $r - 9$. The remaining side has length b where $b^2 = (r + 9)^2 - (r - 9)^2 = 2 \times 18r = 36r$ so $b = 6\sqrt{r}$.



In triangle T_3

$$\left(2\sqrt{r}\right)^2 + (2r - 25)^2 = 25^2$$
$$4r + \left(4r^2 - 4 \times 25r + 25^2\right) = 25^2$$
$$r\left(4 + 4r - 100\right) = 0$$

giving $r = 0$ or 24 and $r = 0$ does not give a circle.
So the radius of C_1 is 24.

Check: $9 + 2r - 25 + 16 = 48$ and $2r = 48$.

S5. If p and q are positive integers, $\max(p, q)$ is the maximum of p and q and $\min(p, q)$ is the minimum of p and q. So for example $\max(3, 6) = 6$ and $\min(3, 6) = 3$.

Determine the number of ordered pairs (x, y) which satisfy the equation

$$\max\big(70, \min(x, y)\big) = \min\big(\max(70, x), y\big)$$

where x and y are positive integers with $x \leqslant 100$ and $y \leqslant 100$.

Solution

(To save space we will abbreviate left-hand side as LHS and right-hand side as RHS.)

If $x \leqslant 70$, then $\min(x, y) \leqslant 70$ so the LHS $= 70$.

Also $\max(70, x) = 70$ so RHS $= \min(70, y) = \text{LHS} = 70$. So $y \geqslant 70$.

If $x > 70$, then $\max(70, x) = x$ so RHS $= \min(x, y)$.

Hence

$$\text{LHS} = \max\big(70, \min(x, y)\big) = \text{RHS} = \min(x, y)$$

$$\min(x, y) \geqslant 70$$

$$y \geqslant 70$$

Check:

If $y < 70$, $\max(70, x) \geqslant 70$ so RHS $= \min(70, y) = y$.

Also LHS $= \max(70, \text{less than } 70) = 70 = \text{RHS} = y$, so $y = 70$ contradicting the original supposition that $y < 70$.

For every $x = 1, 2, \ldots, 100$, y can be $70, 71, \ldots, 100$.

100 values of x times 31 values for y gives 3100 possible ordered pairs.