2019-2020 Senior Division: Problems 1 solutions

S1. There are 10 lockers in a row, numbered from 1 to 10. Each locker is to be painted red or blue or green, subject to the following rules:

- two lockers with numbers n and m are painted different colours whenever $n - m$ is odd;
- it is not necessary to use all 3 colours.

In how many different ways can the row of lockers be painted? Justify your answer.

Solution

If the first locker is red, none of the even lockers can be red. And if another odd locker is green, none of the even lockers can be green. This means that all the even lockers must be blue. So we can paint all the even lockers in one colour and use either or both of the other two colours for the odd lockers, or vice versa.

Case 1

Just use two colours, alternating along the row. There are 3 ways of choosing the first colour and two ways of choosing the second colour, making $3 \times 2 = 6$ ways.

Case 2

Use one colour for the even lockers so 3 choices.

Use the other two colours for the odd lockers, making sure to use both colours. There are 2 choices for each of the 5 odd lockers, giving 25 = 32 possibilities. But this includes the two single colour possibilities, so there are $32 - 2 = 30$ ways of using both colours. So for the whole row there are $3 \times 30 = 90$ ways.

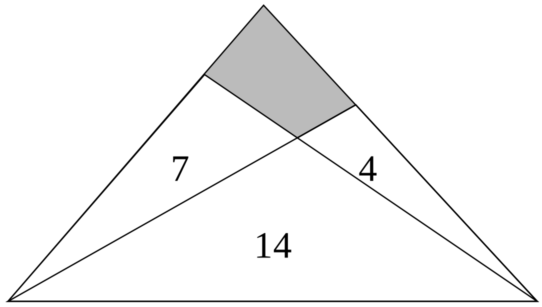
Case 3

Use one colour for the odd lockers and the other two colours for the even lockers, making sure to use both colours. Also 90 ways.

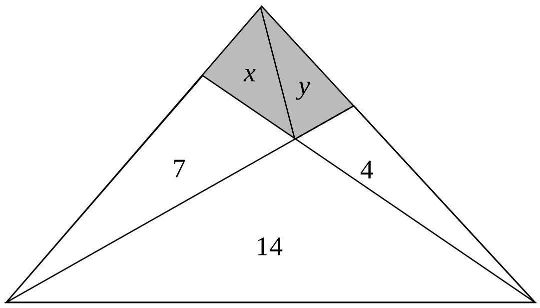
So there are $6 + 90 + 90 = 186$ different ways of painting the row of lockers.

S2. In the diagram (which is not drawn to scale) the small triangles each have the area shown.

Find the area of the shaded quadrilateral.



Solution



Draw in the line from the third vertex of the large triangle to the crossing point within it. Let the areas of the two small triangles formed be x and y as shown.

Triangles with the same height have areas proportional to the lengths of their bases. So

$$\frac{x}{7} = \frac{x + y + 4}{21} \qquad x = \frac{1}{3}(x + y + 4)$$

$$\frac{y}{4} = \frac{x + y + 7}{18} \qquad y = \frac{2}{9}(x + y + 7)$$

$$x + y = \left(\frac{1}{3} + \frac{2}{9}\right)(x + y) + \frac{4}{3} + \frac{14}{9}$$

$$\frac{4}{9}(x + y) = \frac{26}{9} \qquad x + y = \frac{26}{4} = \frac{13}{2}.$$

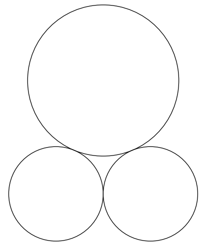
So the shaded area is $6\frac{1}{2}$.

NOTE: This method establishes the whole area but other methods will find x and y and add them.

S3. Two circular discs of radius 5 cm and one circular disc of radius 8 cm are placed flat on a table with their edges touching.

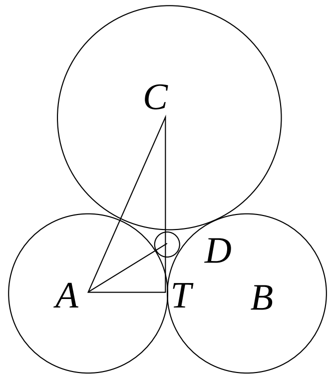
(a) Determine the exact radius of the largest disc that can fit in the space between these three discs.

(b) Determine the exact radius of the smallest disc that can surround these three discs.



Solution

(a) Let A, B, C and D be the centres of the first 4 discs, and let T be the point where discs with centres A and B touch.



Then by symmetry ΔATC is a right angle. AT is 5 cm, AC is $5 + 8 = 13$ cm and hence by Pythagoras $CT = 12$ cm.

Let the radius of the disc centre D be r cm.

So, in triangle ATD, $AT = 5$ cm, $TD = TC - 8 - r = 4 - r$ cm and $AD = 5 + r$ cm

Triangle ATD has a right angle at T. So by Pythagoras,

$$AT^2 + TD^2 = AD^2$$

$$5^2 + (4 - r)^2 = (5 - r)^2$$

$$25 + 16 - 8r + r^2 = 25 + 10r + r^2$$

$$16 = 18r$$

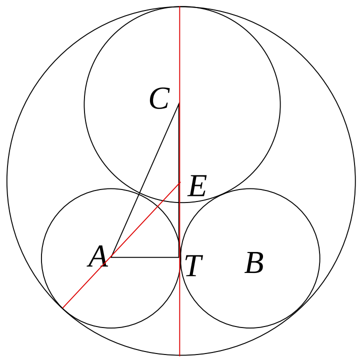
$$r = \frac{16}{18} = \frac{8}{9}.$$

The radius of the fourth disc is 8 cm.

(b) Let E be the centre of the fifth disc and R cm its radius.

$TE = TC + 8 - R = 12 + 8 - R = 20 - R$ and $AE = R - 5$.

Triangle ATE has a right angle at T . So by Pythagoras,



$$AT^2 + TE^2 = TE^2$$

$$5^2 + (20 - R)^2 = (R - 5)^2$$

$$5^2 + 20^2 - 40R + R^2 = R^2 - 10R + 5^2$$

$$400 = 40R - 10R$$

$$R = \frac{400}{30} = \frac{40}{3}.$$

Hence the radius of the fifth disc is $13\frac{1}{3}$ cm.

S4. Find all solutions of the pair of equations

$$x^2 + x^2y^2 + x^2y^4 = 525$$

and

$$x + xy + xy^2 = 35.$$

Solution

$$x^2\left(1 + y^2 + y^4\right) = 525 \qquad (1)$$

$$x\left(1 + y + y^2\right) = 35 \qquad (2)$$

Note that $\left(1 + y + y^2\right) = \frac{3}{4} + \left(\frac{1}{2} + y\right)^2 > 0$, so we are able to divide by $1 + y + y^2$.
Now square (2) and divide to eliminate x

$$\frac{1 + y^2 + y^4}{(1 + y + y^2)^2} = \frac{525}{35^2} = \frac{105}{7 \times 35} = \frac{3}{7}$$

$$7\left(1 + y^2 + y^4\right) = 3\left(1 + y + y^2\right)^2$$

But $\left(1 + y^2 + y^4\right) = 1 + 2y^2 + y^4 - y^2 = \left(1 + y^2\right)^2 - y^2 = \left(1 + y + y^2\right)\left(1 - y + y^2\right)$.
Using this, we have

$$7\left(1 + y^2 + y^4\right) = 3\left(1 + y + y^2\right)^2$$

$$7\left(1 + y + y^2\right)\left(1 - y + y^2\right) = 3\left(1 + y + y^2\right)^2$$

$$7\left(1 - y + y^2\right) = 3\left(1 + y + y^2\right)$$

$$4y^2 - 10y + 4 = 0$$

$$2y^2 - 5y + 2 = 0$$

$$(2y - 1)(y - 2) = 0$$

$$y = \frac{1}{2} \text{ or } 2$$

For $y = \frac{1}{2}$, $x\left(1 + \frac{1}{2} + \frac{1}{4}\right) = 35$ giving $x = 20$.

Check: 400 (1 + 1 + 16) = 400 + 100 + 25 = 525 as required.

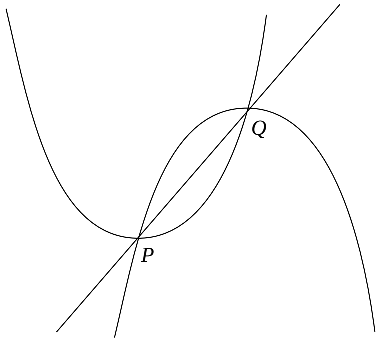
For $y = 2$, $x\left(1 + 2 + 2^2\right) = 35$ giving $x = 5$.

Check: 25 (1 + 4 + 16) = 25 + 100 + 400 = 525 as required.

So the solutions are $x = 5$, $y = 2$ and $x = 20$, $y = \frac{1}{2}$.

S5. The parabola $y = ax^2 + bx + c$ has vertex P and the parabola $y = -x^2 + dx + e$ has vertex Q, where P and Q are distinct points. The two parabolas also intersect at P and Q.
(a) Prove that $2(e - c) = bd$.
(b) Prove that the line through P and Q has slope $\frac{1}{2}(b + d)$ and y-intercept $\frac{1}{2}(c + e)$.

Solution



(a) The vertex of the first parabola is at x-coordinate $-\frac{1}{2}b$.
Since the second parabola also passes through this point

$$\frac{b^2}{4} + b\left(-\frac{b}{2}\right) + c = -\frac{b^2}{4} + d\left(-\frac{b}{2}\right) + e$$

$$\frac{db}{2} = e - c$$

$$2(e - c) = bd$$

(b) $P\left(-\frac{1}{2}b, -\frac{1}{4}b^2 + c\right)$
The vertex of the second parabola is at x-coordinate $\frac{1}{2}d$. So

$$Q = \left(\frac{d}{2}, \frac{d^2}{4} + e\right).$$

So the slope of the line joining P and Q is

$$\begin{aligned} &\frac{\frac{1}{4}d^2 + e - \left(-\frac{1}{4}b^2 + c\right)}{\frac{1}{2}d - \left(-\frac{1}{2}b\right)} \\ &= \frac{d^2 + b^2 + 4e - 4c}{2d + 2b} \\ &= \frac{d^2 + b^2 + 2bd}{2d + 2b} \\ &= \frac{(d + b)^2}{2d + 2b} \\ &= \frac{d + b}{2}. \end{aligned}$$

The equation of the line is $y = \frac{1}{2}(d + b)x + i$. where i is the intercept.
The line passes through P, so

$$-\frac{b^2}{4} + c = \frac{d + b}{2}\left(-\frac{1}{2}b\right) + i$$

$$c = -\frac{db}{4} + i$$

$$i = c + \frac{db}{4} = c + \frac{e - c}{2} + \frac{c + e}{2}$$

i.e. the y-intercept of the line is $\frac{1}{2}(c + e)$ as required.

Neat solution
The equations of the two parabolas are $y = x^2 + bx + c$ and $y = -x^2 + dx + e$.
Adding these two equations, we obtain $2y = (b + d)x + (c + e)$ or
$y = \frac{1}{2}(b + d)x + \frac{1}{2}(c + e)$. The last equation is the equation of a line.

Points P and Q, whose coordinates satisfy the equation of each parabola, must satisfy the equation of the line, and so lie on the line.

But the line through P and Q is unique, so this is the equation of the line through P and Q.

Therefore, the line through P and Q has slope $\frac{1}{2}(b + d)$ and y-intercept $\frac{1}{2}(c + e)$.