2018-2019 Senior Division: Solutions to Problems 2

S1. Show that the product of four consecutive odd integers is always 16 less than a square number.

Deduce that the product of four consecutive odd integers can never be a square number except in one particular case.

Solution

Suppose the four integers are $k - 3,\ k - 1,\ k + 1,\ k + 3$ for some even integer k. Their product can be written

$$(k - 1)(k + 1)(k - 3)(k + 3)$$

$$= \left(k^2 - 1\right)\left(k^2 - 9\right) = k^4 - 10k^2 + 9 = \left(k^2 - 5\right)^2 - 16 \text{ (by completing the square).}$$

This proves the first part.

We inspect the sequence of squares 0, 1, 4, 9, 16, 25, 36, 49, 64, 81, 100,…

There are 2 pairs that differ by 16: 0 and 16, and 9 and 25. There cannot be any more as the difference between further successive squares is greater than 16.

0 cannot be a product of odd integers.

So the only situation where the product of four consecutive odd integers is a square is when the product is 9 and the odd integers are $-3,\ -1,\ 1,\ 3$.

S2. A cardboard box manufacturer makes open-topped boxes which are cubes. Because of changes in the market, there are plans to double the volume of the boxes which are made. The regular supplier of raw cardboard offers a 37.5% discount on the price that was originally being charged. A new supplier offers a deal in which the manufacturer would be paying exactly the same for the raw material for his bigger boxes as was paid for the smaller boxes.

Which is the best deal for the manufacturer?

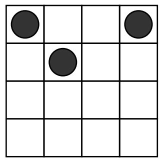
Solution

Let the length of a side of the original box be a metres. So its volume is a^3 m^3 and the amount of cardboard required is $5a^2$ m^2. If the regular supplier charged £x per square metre, the raw material for each box costs £$5a^2x$.

He now doubles the volume of his boxes to $2a^3$ m^3. If the length of the side is A metres then $A^3 = 2a^3$ so that $A = \sqrt[3]{2}a$. The amount of cardboard required for the bigger box is $5A^2 = 5\sqrt[3]{2^2}a^2$ m^2.

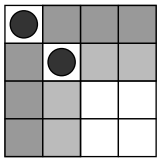
So the regular supplier charges $5\sqrt[3]{2^2}a^2 \times \frac{5}{8}x$ and the new supplier charges £$5a^2x$. Since $\sqrt[3]{2^2} \times \frac{5}{8} < 1$ he should stick with the original supplier.

S3. In a 4 $\times$ 4 grid as shown, place three coins randomly in different squares.



Determine the probability that no two coins lie in the same row or column.

Solution



The first coin can be placed in any square and there are always 6 squares in the same row or column − darker shading in the diagram.
The second coin can be placed in any of the 15 remaining squares, but only $15 - 6 = 9$ of these are in a different row and a different column. So the probability that these coins do not lie in the same row or column is $1 \times \dfrac{9}{15}$.

The third coin can be placed in any of the 14 remaining squares, but only 4 of these are in a different row and a different column − unshaded and unoccupied in the diagram. So the probability that all three coins do not lie in the same row or column is $1 \times \dfrac{9}{15} \times \dfrac{4}{14} = \dfrac{6}{35}$.

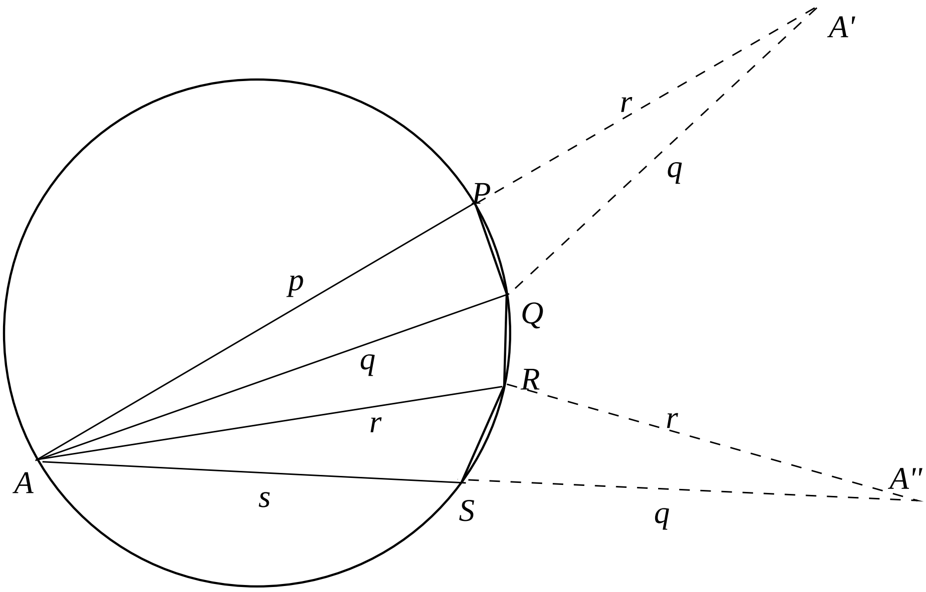
S4. Distinct points A, P, Q, R and S lie on the circumference of a circle and AP, AQ, AR and AS are chords with the property that

$$\angle PAQ = \angle QAR = \angle RAS.$$

Prove that

$$AR(AP + AR) = AQ(AQ + AS).$$

Solution 1



Let the chords AP, AQ, AR, AS have lengths p, q, r, s.

The chords PQ, QR, RS all subtend equal angles in the same circle and therefore are all of the same length.

Angles APQ and ARQ are supplementary, since they are in opposite segments.

So when triangle QAR is rotated about Q until QR is coincident with QP, APA' forms a straight line and the triangle AQA' is isosceles.

Similarly, angles ASR and AQR are supplementary, since they are in opposite segments.

So when triangle QAR is rotated about R until RQ is coincident with RS, ASA'' forms a straight line and the triangle ARA'' is isosceles.

These isosceles triangles are similar, since they have the same base angle. Hence

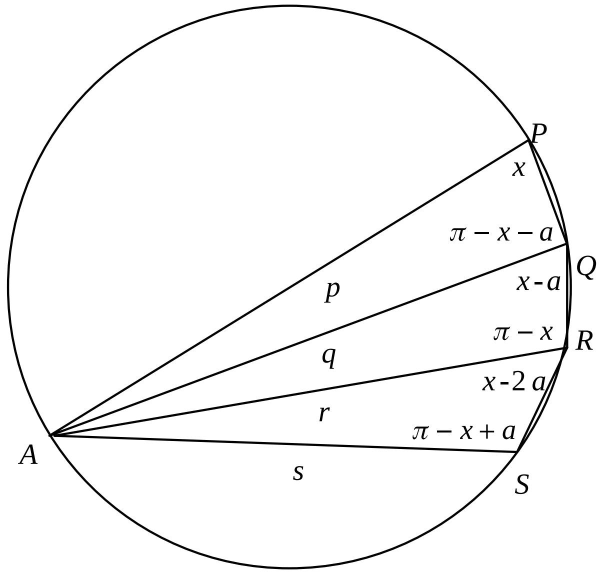
$$\frac{p + r}{q} = \frac{q + s}{r}$$

$$r(p + r) = q(q + s)$$

i.e. $AR(AP + AR) = AQ(AQ + AS)$ as required.

Solution 2

Let the equal angles at A be a and angle APQ be x. Let the chords AP, AQ, AR, AS have lengths p, q, r, s.



Then the angles shown can easily be deduced knowing that opposite angle in a cyclic quadrilateral are supplementary and angles in a triangle sum to π.

Using the sine rule in triangle APQ:

$$\frac{p}{\sin(\pi - x - a)} = \frac{q}{\sin x} \qquad (1)$$

Using the sine rule in triangle AQR:

$$\frac{q}{\sin(\pi - x)} = \frac{r}{\sin(x - a)}. \qquad (2)$$

Using the sine rule in triangle ARS:

$$\frac{r}{\sin(\pi - x + a)} = \frac{s}{\sin(x - 2a)}. \qquad (3)$$

Since $\sin x = \sin(\pi - x)$ and $\sin(x - a) = \sin(\pi - x + a)$ (1), (2) and (3) are all equal (to k say). Hence from (1), $p = k\sin(\pi - x - a)$, from (2) $q = k\sin(\pi - x)$ and from (3) $r = k\sin(\pi - x + a)$ and also $s = k\sin(x - 2a)$.

So

$$AR(AP + AR) = r(p + r) = k\sin(x - a)\cdot k\left[\sin(\pi - x - a) + \sin(x - a)\right]$$
$$= k^2\sin(x - a)\left[\sin(x + a) + \sin(x - a)\right] = k^2\sin x\cdot\tfrac{1}{2}\sin x\cos a$$

and

$$AQ(AQ + AS) = q(q + s) = k\sin x\cdot k\left[\sin x + \sin(x - 2a)\right] = k^2\sin x\cdot\tfrac{1}{2}\sin x\cos a$$

i.e. $AR(AP + AR) = AQ(AQ + AS)$ as required.

S5. In a magic square, the numbers in each row, the numbers in each column, and the numbers on each diagonal have the same sum. Given the magic square shown with all of a, b, c, x, y, z positive, determine the product xyz in terms of a, b and c.

$\log a$	$\log b$	$\log x$
p	$\log y$	$\log c$
$\log z$	q	r

Solution

Note that all of a, b, c, x, y, z are non-zero and positive, so we can always divide or find square roots as necessary.

The totals of the row and column through p are the same, and so

$$\log a + \log z = \log y + \log c$$

Hence

$$\log az = \log yc$$

$$az = yc$$

$$z = \frac{yc}{a} \tag{i}$$

The totals of the row and diagonal through $\log x$ are the same, and so

$$\log a + \log b = \log z + \log y$$

Hence

$$\log ab = \log yz$$

$$ab = yz \tag{ii}$$

$$z = \frac{ab}{y} \tag{iii}$$

Thus from (i) and (iii)

$$z = \frac{yc}{a} = \frac{ab}{y}$$

and

$$y^2 = \frac{a^2b}{c} \qquad \text{or} \qquad y = \frac{a\sqrt{b}}{\sqrt{c}} \tag{iv}$$

The totals of the diagonal and column through r are the same, and so

$$\log a + \log y = \log x + \log c$$

Hence

$$\log ay = \log xc$$

$$x = \frac{ay}{c} \tag{v}$$

Multiply (ii) by x:

$$xyz = xab$$

and on the RHS substitute x from (v) and then y by (iv):

$$\begin{aligned} xyz &= \frac{ay}{c}ab \\ &= \frac{a^2by}{c} \\ &= \frac{a^2b}{c}\frac{a\sqrt{b}}{\sqrt{c}} \\ &= a^3\sqrt{\frac{b^3}{c^3}}. \end{aligned}$$