2018-2019 Senior Division: Problems 1 solutions

S1. Three expert logicians played a game with a set of 21 cards each with a different two-digit prime number. Each drew a card and held it up so that they could not see the number on their own card but could see the number on the cards of each of the others. Ali, Bobby and Charlie in turn were then asked two questions, namely "Is your number the smallest of the three?" and "Is your number the largest of the three?". In the first round all three answered "Don't know" to both questions. The same happened in rounds two and three. In round 4 Ali answered "Don't know" to the first question. What did Ali answer to the second question and what numbers did Bobby and Charlie have?

Solution

There are exactly 21 two-digit primes:

11	13	17	19	23	29	31	37	41	43	
47	53	59	61	67	71	73	79	83	89	97

It is only the order which is important, so consider the numbers 1, 2, ... 21 instead.

Round 1

If A could see the 1 card, she would know she had a number greater than 1 and would answer no to the first question. So B and C cannot have 1.

If B could see the 1 card or the 2 card, he would know he had a number greater than 2 and would answer no to his first question. So A cannot have 1 or 2 and C cannot have 2.

If C could see the 2 card or the 3 card, she would know she had a number more than 3 and would answer no to the first question. So A cannot have 3 and B cannot have 2 or 3.

If A could see the 21 card, she would know she had a number less than 21 and would answer no to the second question. So B and C cannot have 21.

If B could see the 21 card or the 20 card, he would know he had a number less than 20 and would answer no to his second question. So A cannot have 21 or 20 and C cannot have 20.

If C could see the 20 card or the 19 card, she would know she had a number less than 19 and would answer no to the second question. So A cannot have 19 and B cannot have 20 or 19.

Rounds 2 and 3 are similar. The numbers eliminated each time are:

round	question	A	B	C
1	A1		1	1
	B1	1, 2		2
	C1	3	2, 3	
2	A1		4	3, 4
	B1	4, 5		5
	C1	6	5, 6	
3	A1		7	6, 7
	B1	7, 8		8
	C1	9	8, 9	
1	A2		21	21
	B2	21, 20		20
	C2	19	20, 19	
2	A2		18	19, 18
	B2	18, 17		17
	C2	16	17,16	
3	A2		15	16, 15
	B2	15, 14		14
	C2	13	14,13	

So A can only have 10, 11 or 12, B can only have 10, 11 or 12 and C can only have 9, 10, 11, 12 or 13.

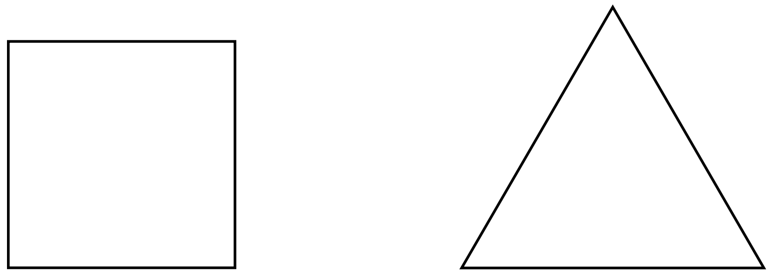
A can't see 9 or 10, because then she would know to answer no to the next question. And she can't see both 11 and 12 or both 12 and 13 because then she would answer yes.

So she must be able to see 11 and 13.

Hence Ali would answer no to the question "Is your number the largest of the three?". And Bobby has the 11th prime, 47 and Charlie has the 13th prime, 59.

S2. Consider a square with side 15 cm and an equilateral triangle with the same perimeter. Which has the greater area? And by how much?

Solution

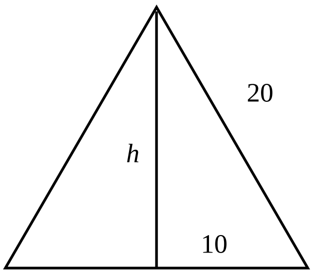


The area of the square is $15 \text{ cm} \times 15 \text{ cm} = 225 \text{ cm}^2$ and its perimeter is $4 \times 15 \text{ cm} = 60 \text{ cm}$.

So each side of the triangle is 20 cm. We will calculate the area of the triangle in three ways:

Method 1:

Put in an altitude. This bisects the base and we can use Pythagoras' Theorem:



$$h^2 + 10^2 = 20^2$$

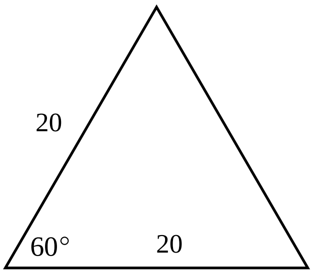
$$h^2 = 400 - 100 = 300$$

$$h = \sqrt{300}$$

So the area is $\frac{1}{2} \times 20 \times \sqrt{300} = 100\sqrt{3} \text{ cm}^2$.

Method 2:

We can use trigonometry. The area is given by



$$\tfrac{1}{2} \times 20 \times 20 \times \sin 60°$$

$$= 10 \times 20 \times \frac{\sqrt{3}}{2}$$

$$= 100\sqrt{3} \text{ cm}^2$$

Method 3:

We can use Heron's Formula which states that the area of a triangle is given by

$$\sqrt{s(s - a)(s - b)(s - c)}$$

where the sides are a, b, c and s is the semi-perimeter, $\frac{1}{2}(a + b + c)$. So the area is

$$\sqrt{30 \times 10 \times 10 \times 10} = \sqrt{3 \times 100 \times 100} = 100\sqrt{3} \text{ cm}^2$$

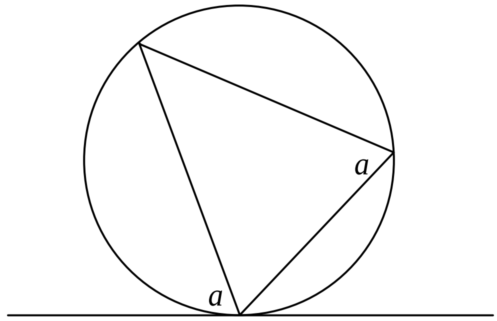
Thus the area of the square is 225 cm^2, the area of the triangle is $100\sqrt{3} \text{ cm}^2$ and the area of the square is bigger since

$$100\sqrt{3} < 100\sqrt{4} = 200 < 225.$$

And the area of the square is bigger than the area of the triangle by

$$225 - 100\sqrt{3} = 25\left(9 - 4\sqrt{3}\right) \approx 25 \times (9 - 6{\cdot}9) \approx 50 \text{ cm}^2.$$

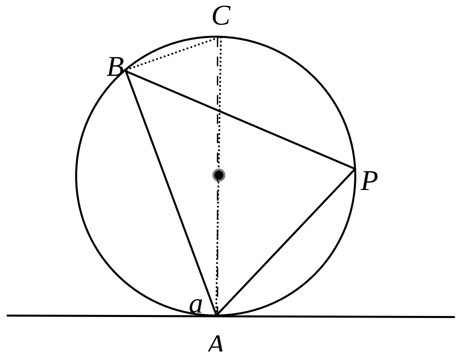
S3. (i) Prove the alternate segment theorem, which states that the angle between the tangent and chord at the point of contact is equal to the angle in the alternate segment.

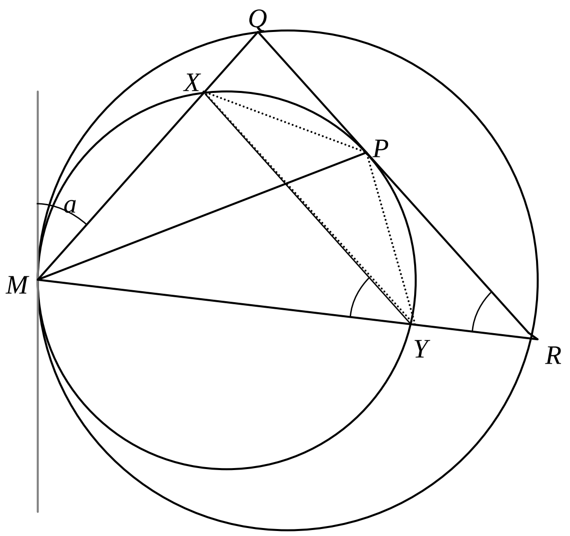


(ii) Two circles touch internally at M. A straight line touches the inner circle at P and cuts the outer circle at Q and R.
Prove that $\angle QMP = \angle RMP$.

Solution

(i) Consider the chord AB which makes angle a with the tangent at A.
The diameter of the circle through A is at right angles to the tangent at A, so $\angle BAC = 90° - a$.
The angle in a semicircle is 90°, so
$\angle BCA = 180° - 90° - (90° - a) = a$.
Finally, angles in the same segment are equal, so
$\angle BPA = a$ as required.





(ii) Let the angle between QM and the tangent be a. Then

$\angle XYM = a$	(angle in the opposite segment to chord XM in the small circle)
$\angle QRM = a$	(angle in the opposite segment to chord QM in the large circle)
So XY // QR	(corresponding angles equal)

$\angle QMP = \angle XMP$	
$\angle XMP = \angle XYP$	(angles in same segment)
$\angle XYP = \angle YPR$	(alternate angles)
$\angle YPR = \angle YMP$	(alternate segment theorem)
$\angle YMP = \angle RMP$	

Hence $\angle QMP = \angle RMP$ as required.

S4. Find all the positive integers k for which $7 \times 2^k + 1$ is a perfect square.

Solution

Suppose that $7 \times 2^k + 1 = n^2$ for some positive integer n. Then

$$7 \times 2^k = n^2 - 1 = (n + 1)(n - 1),$$

and so 7 divides either $(n + 1)$ or $(n - 1)$.

Also, $n + 1$ and $n - 1$ have the same parity and, as their product is even, they are both even.

Suppose first that 7 divides $n + 1$. Then $n + 1 = 14m$ for some positive integer m, giving

$$7 \times 2^k = 14m(14m - 2)$$

and then

$$2^k = 4m(7m - 1),$$

so $4m$ and $7m - 1$ are both powers of 2. However, this cannot happen: if $m = 1$, $7m - 1 = 6$ whilst, if m is a positive power of 2, $7m - 1$ is odd.

Thus 7 cannot divide $n + 1$ and so it must divide $n - 1$. Hence $n - 1 = 14m$ for some positive integer m, giving

$$7 \times 2^k = 14m(14m + 2)$$

and, in this case,

$$2^k = 4m(7m + 1).$$

Then $4m$ and $7m + 1$ are both powers of 2. In much the same way as above, m cannot be a positive power of 2 (as $7m + 1$ would then be odd). However, $m = 1$ and $7m + 1 = 8$ is possible. This gives $2^k = 32$, i.e. $k = 5$, and then $7 \times 2^5 + 1 = 7 \times 32 + 1 = 225 = 15^2$. Thus $k = 5$ is the only possible value of k.

S5. My husband and I recently attended a party at which there were four other married couples. No one shook hands either with themself or with their spouse and no one shook hands with the same person more than once. After all the handshakes were over, I asked each person, including my husband, how many hands they had shaken. To my surprise each gave a different answer. How many hands did my husband shake?

Solution 1

Initially there are 5 couples. Someone shakes 8 hands − every other person except their spouse. So their spouse is the only person who can shake 0 hands. Eliminate this couple, so there are now 4 couples left.

Someone shakes a further 6 hands (7 shakes in all) − every remaining person except their spouse. So their spouse is the only person who can shake 0 further hands (1 shake in total). Eliminate this couple, so there are now 3 couples left.

Someone shakes a further 4 hands (6 shakes in all) - every remaining person except their spouse. So their spouse is the only person who can shake 0 further hands (2 shakes in total). Eliminate this couple, so there are now 2 couples left.

Someone shakes a further 2 hands (5 shakes in all) - every remaining person except their spouse. So their spouse is the only person who can shake 0 further hands (3 shakes in total). Eliminate this couple, so there is now 1 couple left.

The remaining couple have each shaken 4 hands. So they must be me and my husband, since everyone I asked had shaken a different number of hands.
My husband shook 4 hands.

Solution 2

Let ‘my husband’ be denoted by M_1 and ‘me’ by M_2.
Let the other couples be denoted by A_1, A_2, B_1, B_2, C_1, C_2, D_1, D_2.

The maximum number of hands which can be shaken by anyone is 8 so the possible answers are 0, 1, 2, 3, 4, 5, 6, 7, 8. But nine people answer. So the answers are precisely 0, 1, 2, 3, 4, 5, 6, 7, 8.

If M_1 answers 8, he has shaken hands with all others (always exclude, of course, M_2), so each of them has shaken hands at least once and no-one can answer 0. Thus M_1 cannot answer 8.
Let A_1 answer 8. So we have the shakes (A_1, B_1), (A_1, B_2), (A_1, C_1), (A_1, C_2), (A_1, D_1), (A_1, D_2), (A_1, M_1), (A_1, M_2).

If M_1 answers 7, he has shaken hands with all others except A_2, so all others (excluding A_2) have shaken at least two hands and no-one can answer 1. So M_1 cannot answer 7.
Let B_1 answer 7. So we have shakes (B_1, A_1), (B_1, C_1), (B_1, C_2), (B_1, D_1), (B_1, D_2), (B_1, M_1), (B_1, M_2).
So B_2 answers 1 and we have shakes (B_2, A_1).

If M_1 answers 6, then he has shaken hands with all others except A_2 and B_2, so all others have shaken at least three hands and none can answer 2. So M_1 does not answer 6.
Let C_1 answer 6. This gives the shakes (C_1, A_1), (C_1, B_1), (C_1, D_1), (C_1, D_2), (C_1, M_1), (C_1, M_2). So C_2 answers 2 and has shakes (C_2, A_1), (C_2, B_1).

If M_1 answers 5, he has shaken hands with all others except A_2, B_2, C_2, so all others have shaken at least 4 hands and no-one can answer 3. So M_1 does not answer 5.
Let D_1 answer 5 giving the shakes (D_1, A_1), (D_1, B_1), (D_1, C_1), (D_1, M_1), (D_1, M_2).
So D_2 answers 3 and has shakes (D_2, A_1), (D_2, B_1), (D_2, C_1).

So M_1 answers 4.