Senior Division 2017-2018 Round 2 Solutions

S1. Goliath and David play a game in which there are no ties. Each player is equally likely to win each game. The first player to win 4 games becomes the champion, and no further games are played. Goliath wins the first two games. What is the probability that David becomes the champion?

Solution

Since the first player to win 4 games becomes the champion, Goliath and David play at most 7 games. (The maximum number of games comes when the two players have each won 3 games and then one player becomes the champion on the next (7th) game.) We are told that Goliath wins the first two games.

For David to become the champion, the two players must thus play 6 or 7 games, because David wins 4 games and loses at least 2 games. We note that David cannot lose 4 games, otherwise Goliath would become the champion.

If David wins and the two players play a total of 6 games, then the sequence of wins must be GGDDDD. (Here D stands for a win by David and G stands for a win by Goliath.)

If David wins and the two players play a total of 7 games, then David wins 4 of the last 5 games and must win the last (7th) game since he is the champion.

Therefore, the sequence of wins must be GGGDDDD or GGDGDDD or GGDDGDD or GGDDDGD. (In other words, Goliath can win the 3rd, 4th, 5th, or 6th game.)

The probability of the sequence GGDDDD occurring after Goliath has won the first 2 games is

$$\left(\frac{1}{2}\right)^4 = \frac{1}{16}.$$

This is because the probability of a specific outcome in any specific game is $\frac{1}{2}$, because each player is equally likely to win each game, and there are 4 games with undetermined outcome.

Similarly, the probability of each of the sequences GGGDDDD, GGDGDDD, GGDDGDD, and GGDDDGD occurring is

$$\left(\frac{1}{2}\right)^5 = \frac{1}{32}.$$

Therefore, the probability that Goliath wins the first two games and then David becomes the champion is

$$\frac{1}{16} + 4 \times \frac{1}{32} = \frac{3}{16}.$$

S2. Let n be a three-digit number and let m be the number obtained by reversing the order of the digits in n. Suppose that m does not equal n and that $n + m$ and $n - m$ are both divisible by 7. Find all such pairs n and m.

Solution

Since $(n + m) + (n - m) = 2n$ and $(n + m) - (n - m) = 2m$, 7 divides both $2n$ and $2m$; hence 7 divides both n and m. Let $n = 100a + 10b + c$; then $m = 100c + 10b + a$.

We can assume, by interchanging n and m if necessary, that $a > c$ (noting that $a \neq c$). Since $100 = 7 \times 14 + 2$ and $10 = 7 + 3$, we have

$$n = (14 \times 7 + 2)a + (7 + 3)b + c$$

$$= 7 \times (14a + b) + (2a + 3b + c),$$

and, since n is divisible by 7, so is $2a + 3b + c$. Similarly, $2c + 3b + a$ is also divisible by 7. Subtracting these,

$$(2a + 3b + c) - (2c + 3b + a) = a - c,$$

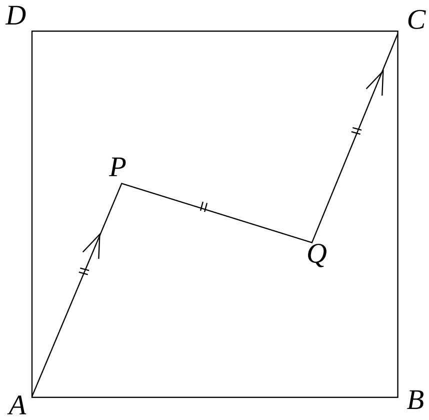
so we deduce that $a - c$ is divisible by 7. Since a and c are integers between 0 and 9 and, by arrangement, $a > c$, $a - c = 7$. There are only three possibilities:

either $a = 8$ and $c = 1$; or $a = 9$ and $c = 2$; or $a = 7$ and $c = 0$.

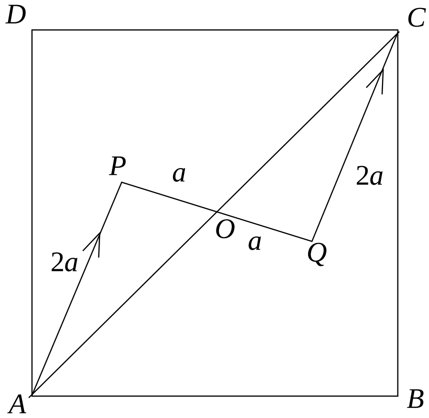
Since $2a + 3b + c$ is divisible by 7, the first case gives $3b + 17$ is divisible by 7 and hence $3b + 3 = 3(b + 1)$ is as well, giving $b = 6$. In the second case, $3b + 20$ is divisible by 7 and hence $3b - 1$ is as well, giving $b = 5$. In the third case, b must be divisible by 7, so $b = 0$ or $b = 7$.

Thus the only possible pairs of numbers are $\{861, 168\}$, $\{952, 259\}$, $\{700, 007\}$ or $\{770, 077\}$.

S3. $ABCD$ is a square. Points P and Q lie within the square such that AP, PQ and QC are all the same length and AP is parallel to QC.
Determine the minimum possible size of $\angle DAP$.



Solution
Let AP, QC and PQ have length $2a$ units and let O be the point where PQ and AC cross.



Then $\angle APO = \angle CQO$ (alternate angles).
And $\angle PAO = \angle QCO$ (alternate angles).
So triangles APO and CQO are congruent (AAS).
Hence $PO = OQ = a$.

By symmetry, O is also the midpoint of the square. Hence $\angle DAO = 45°$.

Let $AO = 1$ unit.
Then using the cosine rule in triangle APO

$$a^2 = (2a)^2 + 1^2 - 2 \times 2a \times 1 \times \cos\angle PAO.$$

$$\cos\angle PAO = \frac{3a^2 + 1}{4a} = \frac{1}{4}\left(3a + \frac{1}{a}\right).$$

To minimise $\angle DAP$, we need to maximise $\angle PAO$ i.e. minimise $\cos\angle PAO$.
Differentiating with respect to a and setting the derivative equal to 0:

$$\frac{1}{4}\left(3 - \frac{1}{a^2}\right) = 0$$

$$a = \frac{1}{\sqrt{3}}$$

The second derivative is positive, so this is a local minimum.

$$\cos\angle PAO = \frac{\sqrt{3}}{2}.$$

So the maximum size of $\angle PAO$ is 30° (and $\angle POA$ is a right angle).
Hence the minimum size of $\angle DAP$ is $45° - 30° = 15°$.

S4. Determine all values of x for which

$$\left(\sqrt{x}\right)^{\log_{10} x} = 100.$$

Solution

Let $y = \sqrt{x}$. Then $x = y^2$ and

$$y^{\log_{10} y^2} = y^{2\log_{10} y} = 100.$$

Taking square roots of both sides:

$$y^{\log_{10} y} = 10.$$

Let $z = \log_{10} y$. Then $y = 10^z$ and

$$\begin{aligned} \left(10^z\right)^z &= 10^{z^2} = 10 \\ z^2 &= 1 \\ z &= \pm 1 \end{aligned}$$

When $z = 1$, $y = 10$ and $x = 100$.

When $z = -1$, $y = \dfrac{1}{10}$ and $x = \dfrac{1}{100}$.

So the only values of x are 100 and $\dfrac{1}{100}$.

(Check by substitution that these values are solutions.)

Solution 2: Using rules for manipulating logarithms

$$\begin{aligned} \left(\sqrt{x}\right)^{\log_{10} x} &= 100 \\ \log_{10}\left(\left(\sqrt{x}\right)^{\log_{10} x}\right) &= \log_{10} 100 \\ \left(\log_{10} x\right)\left(\log_{10} \sqrt{x}\right) &= 2 \\ \left(\log_{10} x\right)\left(\log_{10} x^{\frac{1}{2}}\right) &= 2 \\ \left(\log_{10} x\right)\left(\tfrac{1}{2}\log_{10} x\right) &= 2 \\ \left(\log_{10} x\right)^2 &= 4 \\ \log_{10} x &= \pm 2 \\ x &= 10^{\pm 2} \end{aligned}$$

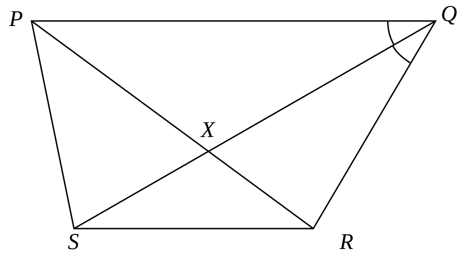
$$\text{Therefore, } x = 100 \text{ or } x = \frac{1}{100}.$$

S5. In a quadrilateral $PQRS$, the sides PQ and SR are parallel, and the diagonal QS bisects angle PQR. Let X be the point of intersection of the diagonals PR and QS.

Prove that $\dfrac{PX}{XR} = \dfrac{PQ}{QR}$.

In a triangle ABC the lengths of all three sides are positive integers. The point M lies on the side BC so that AM is the internal bisector of the angle BAC. Also, $BM = 2$ and $MC = 3$.
What are the possible lengths of the sides of the triangle ABC?

Solution



$$\angle PQS = \angle QSR \qquad \text{(alternate angles)}$$

Hence

$$\angle PQX = \angle XSR$$

$$\angle QPR = \angle PRS \qquad \text{(alternate angles)}$$

Hence

$$\angle QPX = \angle XRS$$

and so triangles PQX and RSX are similar.

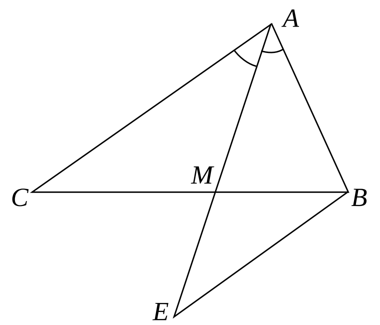
So

$$\frac{PX}{RX} = \frac{PQ}{RS}$$

Also $\angle SQR = \angle PQS = \angle QSR$, so triangle RQS is isosceles and $RS = RQ$.

Thus

$$\frac{PX}{XR} = \frac{PQ}{QR}.$$



Let BE be parallel to AC and meet AM produced at E.
Then the diagram is similar to the one above slightly rotated and hence

$$\frac{CM}{MB} = \frac{CA}{AB} = \frac{3}{2} \Rightarrow CA = \frac{3}{2}AB.$$

Since the sides are all integers, $AB = 2k$, $CA = 3k$ and $BC = 5$ for $k = 1,2,3, \ldots$
But the sides must form a physical triangle.
When $k = 1$ the triangle vanishes. But $k = 2$ is OK.
As k increases, AC is the longest side, so we require

$$AC < AB + BC$$

$$3k < 2k + 5$$

$$k < 5$$

So the possible triangles have $k = 2,\ 3$ or 4.
i.e. sides $AB = 4$, $CA = 6$ and $BC = 5$
or $AB = 6$, $CA = 9$ and $BC = 5$
or $AB = 8$, $CA = 12$ and $BC = 5$