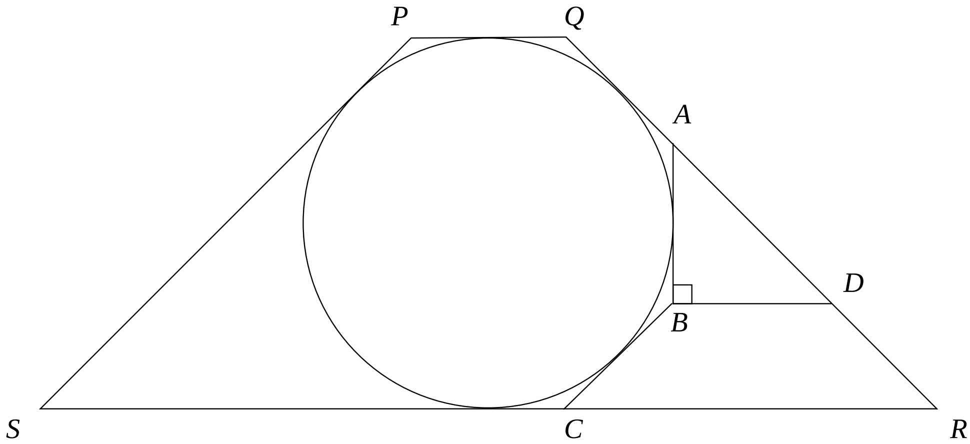
Senior Division 2017-2018 Round 1 Solutions

S1. In a trapezium $PQRS$, PQ is parallel to SR and $\angle SPQ = \angle RQP = 135°$. The trapezium contains an inscribed circle and the length of PQ is 1 cm. What is the **exact** length of QR?

Solution 1



A regular octagon contains an inscribed circle which must touch all sides. The angle between adjacent sides of a regular octagon is 135° and all pairs of opposite sides are parallel.

The trapezium $PQRS$ has an inscribed circle touching all its sides. The angle between adjacent sides, SP and PQ, is 135°. Similarly for $\angle PQR$ between PQ and SR. The opposite sides PQ and QR are given as parallel. Thus the regular octagon with one side PQ and the same inscribed circle as the trapezium $PQRS$ has parts of its other sides coincident with SP, QR and SR as shown in the diagram.

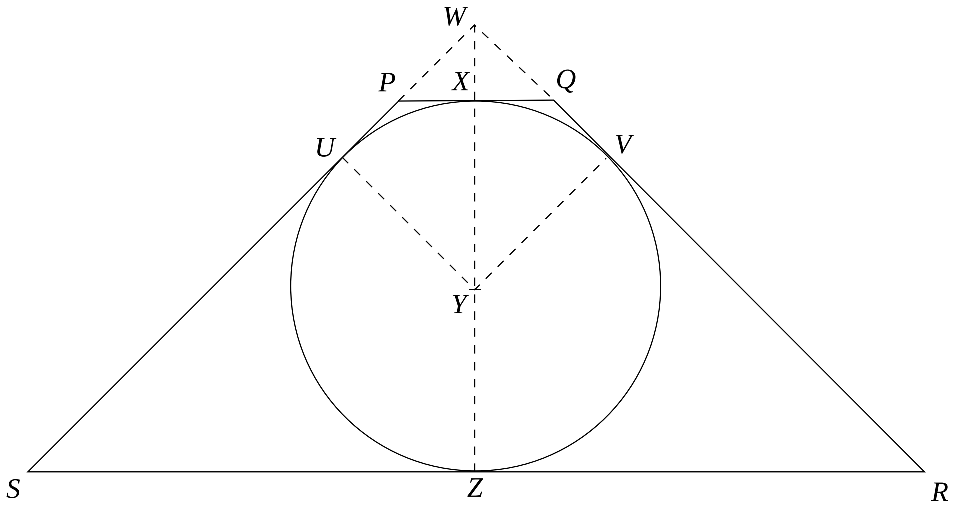
The lines QA, AB and BC are also sides of the regular octagon.

BD is parallel to SR.

$AB = BD = 1$ cm, so $AD = \sqrt{2}$ cm.

So $QR = QA + AD + DR = \left(1 + \sqrt{2} + 1\right)$ cm $= \left(2 + \sqrt{2}\right)$ cm.

S1. *Solution* 2



Extend the sides SP and RQ to meet at W and let X be the midpoint of PQ, Y the centre of the circle and Z the foot of the vertical diameter.

Since $\angle SPQ = 135°$, we know that $\angle QPW = 45°$ and similarly $\angle WQP = 45°$ so $\angle PWQ = 90°$ and WPQ is an isosceles right-angled triangle in which $WP = WQ$ so

$$WP^2 + WQ^2 = PQ^2$$

$$2WP^2 = 1^2 \Rightarrow WP^2 = \frac{1}{2} \Rightarrow WP = \frac{1}{\sqrt{2}}.$$

But X is the midpoint of PQ so $PX = \frac{1}{2}$ and $QX = \frac{1}{2}$.

Now draw in the radii YU and YV. Since SW and RW are tangents $\angle YUW$ and YVW are both 90° so $WVYU$ is a square. Furthermore, since they are equal tangents, $UP = PX = \frac{1}{2}$ and $VQ = QX = \frac{1}{2}$.

So, we have $UW = UP + PW = \frac{1}{2} + \frac{1}{\sqrt{2}} = \frac{1+\sqrt{2}}{2}$. But $WVYU$ is a square so the radii YU and YV are both equal to UW. So we have

$$WZ = WX + XY + YZ = \frac{1}{2} + \left(\frac{1}{2} + \frac{1}{\sqrt{2}}\right) + \left(\frac{1}{2} + \frac{1}{\sqrt{2}}\right) = \frac{3 + 2\sqrt{2}}{2}$$

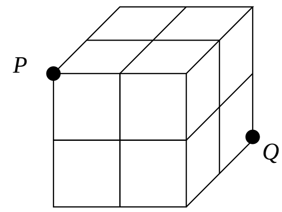
As PQ is parallel to SR we now know that $\angle ZRQ = 45°$ and, from above, $\angle XWQ = 45°$ so triangle WZR is isosceles hence

$$WZ^2 + ZR^2 = WR^2 \Rightarrow WR^2 = 2\left(\frac{3 + 2\sqrt{2}}{2}\right)^2 = \left(\frac{3 + 2\sqrt{2}}{\sqrt{2}}\right)^2$$

Hence

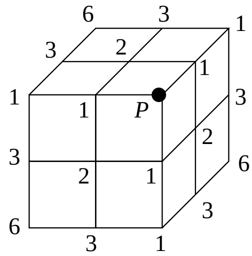
$$QR = WR - WQ = \frac{3}{\sqrt{2}} + 2 - \frac{1}{\sqrt{2}} = \sqrt{2} + 2.$$

S2. Each of the six faces of a solid cube is divided into four squares as indicated in the diagram. Starting from vertex P paths can be travelled to vertex Q along connected line segments. Each movement along a path must take one closer to Q. How many possible paths are there from P to Q?

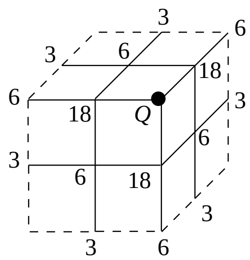


Solution

First count the paths from P towards the edges visible in the diagram below. At each node, the number of paths is the sum of the number of paths to the previous possible nodes.



Then look at the other side of the cube with Q in the centre and continue. Note that the dashed lines have already been traversed.



Thus there are 54 paths from P to Q.

S3. In a magic square, the sum of the numbers in each diagonal, row and column is the same. What is the value of $y + z$ in this 3×3 magic square?

v	24	w
18	x	y
25	z	21

Solution

$$v + x + 21 = v + 18 + 25 \Rightarrow x = 22$$

$$w + x + 25 = w + 24 + v \Rightarrow v = x + 1 = 23$$

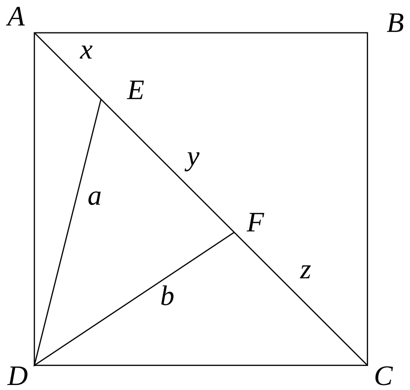
$$\begin{array}{ccc} 23 & 24 & w \\ 18 & 22 & y \\ 25 & z & 21 \end{array}$$

$$z = 66 - 46 = 20; \; w = 66 - 47 = 19; \; y = 66 - 40 = 26$$

$$\begin{array}{ccc} 23 & 24 & 19 \\ 18 & 22 & 26 \\ 25 & 20 & 21 \end{array}$$

$$y + z = 46.$$

S4.

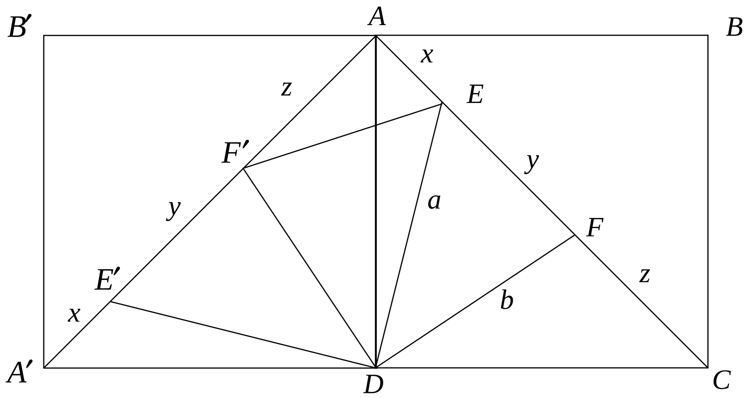


In the diagram, $ABCD$ is a square. Points E and F are chosen on AC so that angle EDF is 45°. If $AE = x$, $EF = y$ and $FC = z$, prove that

$$y^2 = x^2 + z^2.$$

Solution 1

Rotate the square 90° anticlockwise about D to obtain a second square as shown in the diagram below. The new points are indicated with a ′.



Angle EAF' is a right angle since the rotation was through 90°.

So triangle EAF' is a right-angled triangle with sides x and z. We need to show that its hypotenuse is y.

Consider the triangles $F'DE'$ and FDE.

$$\angle F'DA = \angle FDC = 90° - \angle ADE - \angle EDF = 90° - \angle ADE - 45° = 45° - \angle ADE.$$

So

$$\angle F'DE = \angle F'DA + \angle ADE = 45° = \angle FED.$$

Also $F'D = FD$, since $F'D$ is the rotated FD, and DE is common to both triangles.

Hence they are congruent. (SAS)

So the corresponding sides $F'E$ and FE are equal with length y.

Using Pythagoras theorem in the right-angled triangle EAF' we have $y^2 = x^2 + z^2$.

S4. *Solution* 2

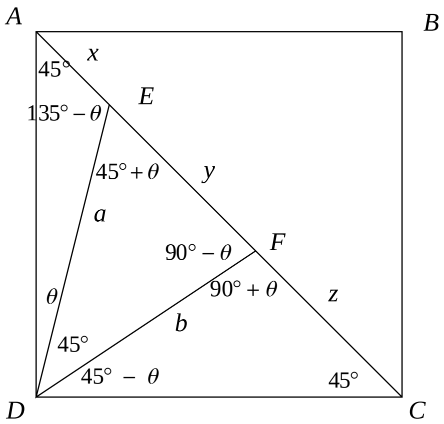
Since AC is a diagonal of square $ABCD$, then $\angle EAD = \angle FCD = 45^\circ$.
Let $\angle ADE = \theta$.
Since the angles in a triangle have a sum of 180°, then

$$\angle AED = 180^\circ - \angle EAD - \angle ADE = 180^\circ - 45^\circ - \theta = 135^\circ - \theta.$$

Since AEF is a straight angle, then

$$\angle DEF = 180^\circ - \angle AED = 180^\circ - (135^\circ - \theta) = 45^\circ + \theta.$$

Continuing in this way, we find that $\angle EFD = 90^\circ - \theta$, $\angle DFC = 90^\circ + \theta$ and $\angle FDC = 45^\circ - \theta$.



Using the Sine Rule in ΔAED, we see that $\dfrac{AE}{\sin \angle ADE} = \dfrac{ED}{\sin \angle EAD}$ or $\dfrac{x}{\sin\theta} = \dfrac{ED}{\sin 45^\circ}$.

Using the Sine Rule in ΔDEF, we see that $\dfrac{EF}{\sin \angle EDF} = \dfrac{ED}{\sin \angle EFD}$ or $\dfrac{y}{\sin 45^\circ} = \dfrac{ED}{\sin (90^\circ - \theta)}$.

Using the Sine Rule in ΔDEF, we see that $\dfrac{EF}{\sin \angle EDF} = \dfrac{FD}{\sin \angle DEF}$ or $\dfrac{y}{\sin 45^\circ} = \dfrac{FD}{\sin (45^\circ + \theta)}$.

Using the Sine Rule in ΔDFC, we see that $\dfrac{FC}{\sin \angle FDC} = \dfrac{FD}{\sin \angle DCF}$ or $\dfrac{z}{\sin (45^\circ - \theta)} = \dfrac{FD}{\sin 45^\circ}$.

Dividing the first of these equations by the second, we obtain $\dfrac{x \sin 45^\circ}{y \sin\theta} = \dfrac{\sin (90^\circ - \theta)}{\sin 45^\circ}$ or $\dfrac{x}{y} = \dfrac{\sin (90^\circ - \theta) \sin\theta}{\sin^2 45^\circ}$.

Dividing the fourth of these equations by the third, we obtain $\dfrac{z \sin 45^\circ}{y \sin (45^\circ - \theta)} = \dfrac{\sin (45^\circ + \theta)}{\sin 45^\circ}$ or $\dfrac{z}{y} = \dfrac{\sin (45^\circ + \theta) \sin (45^\circ - \theta)}{\sin^2 45^\circ}$.

Since $\sin (90^\circ - \alpha) = \cos\alpha$ for every angle α, then $\sin (90^\circ - \theta) = \cos\theta$.
Also $\sin (45^\circ + \theta) = \sin (90^\circ - (45^\circ - \theta)) = \cos (45^\circ - \theta)$.

Using this and the fact that $\dfrac{1}{\sin^2 45^\circ} = \dfrac{1}{(1/\sqrt{2})^2} = 2$, the expressions for $\dfrac{x}{y}$ and $\dfrac{z}{y}$ become

$$\frac{x}{y} = 2 \cos\theta \sin\theta = \sin 2\theta$$

and

$$\frac{z}{y} = 2\cos(45^\circ - \theta)\sin(45^\circ - \theta) = \sin(2(45^\circ - \theta)) = \sin(90^\circ - 2\theta) = \cos 2\theta.$$

Finally, this gives us that

$$\frac{x^2}{y^2} + \frac{z^2}{y^2} = \left(\frac{x}{y}\right)^2 + \left(\frac{z}{y}\right)^2 = \sin^2 2\theta + \cos^2 2\theta = 1.$$

Since $\dfrac{x^2}{y^2} + \dfrac{z^2}{y^2} = 1$, we have $x^2 + z^2 = y^2$, as required.

S5. (a) A class of 15 is to be divided into groups of three for practical work. There is one pair of twin in the class. Show that if the groups are selected at random the probability that the twins are i the same group of three is $\frac{1}{7}$.

(b) If there were two pairs of twins in the class determine the probability that there would be at least one group containing a pair of twins.

Solution

(a)

Represent the 5 groups of three by 5 boxes, each to contain three group members and let the twins be A1 and A2. Place twin A1 in a box, and make it the first box:

spaces	2	3	3	3	3
	A1				

The twins will be together in the same group only if A2 takes one of the available spaces in the first box, out of the total 14 equally likely available spaces. This makes the probability of the twi being in the same group $\frac{2}{14}$ or $\frac{1}{7}$ as required.

(b)

Method 1

Let the first pair of twins be A1 and A2, and the second pair B1 and B2.

P(at least one pair in same group)

= P(pair A together) + P(pair B together) − P(pair A together and pair B together)

$= \frac{1}{7} + \frac{1}{7}$ − P(pair A together and pair B together)

Now find P(pair A together and pair B together).

spaces	1	3	3	3	3
	A1 A2				

First place pair A together in box 1, which has probability $\frac{1}{7}$ as shown above. Then if B1 occupie the remaining space in box 1, with probability $\frac{1}{13}$, none of the possible 12 remaining positions fo B2 allow them to be together. Alternatively, if B1 occupies any of the other 12 out of the 13 spaces available then B2 must take one of the other 2 spaces in the same box, out of the total 12 spaces still available. So

P (pair A together and pair B together) = $\frac{1}{7}\left(\frac{1}{13} \times \frac{0}{12} + \frac{12}{13} \times \frac{2}{12}\right) = \frac{2}{7 \times 13}$.

Hence

P(at least one pair in same group) = $\frac{1}{7} + \frac{1}{7} - \frac{2}{7 \times 13} = \frac{13 + 13 - 2}{7 \times 13} = \frac{24}{91}$.

S5. *Method* 2

Let the first pair of twins be A1 and A2, and the second pair B1 and B2.
P(at least one pair in same group) = 1 − P(both pairs apart)
Now find P(both pairs apart).

spaces	2	2	3	3	3
	A1	A2			

Place pair A apart in the first two boxes, with probability $1 - \frac{1}{7} = \frac{6}{7}$. If B1 occupies one of the 4 remaining spaces in these boxes, with probability $\frac{4}{13}$, then B2 can occupy any except the other space in the same box, i.e. 11 out of 12 remaining spaces. Alternatively, if B1 occupies any of the other 9 out of the 13 spaces available in the empty boxes then B2 cannot take one of the other 2 spaces in the same box, leaving 10 out of the total 12 spaces available for B2. So

P(both pairs apart) = $\frac{6}{7}\left(\frac{4}{13} \times \frac{11}{12} + \frac{9}{13} \times \frac{10}{12}\right) = \frac{6}{7} \times \frac{44 + 90}{13 \times 12} = \frac{67}{91}$.

Hence P(at least one pair in same group) = $1 - \frac{67}{91} = \frac{24}{91}$.