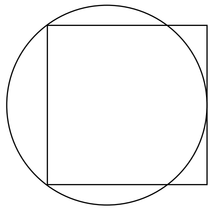
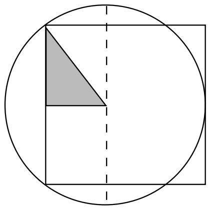
Senior Division 2016-2017 Round 2 Solutions

S1. In the diagram the square has two of its vertices on the circle of radius 1 unit and the other two vertices lie on a tangent to the circle. Find the area of the square.



Solution



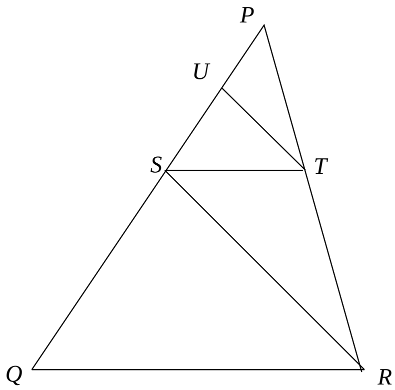
Let the side of the square be $2x$ units.
Then the shaded right-angled triangle shown has sides x and $2x - 1$ and hypotenuse 1.
By Pythagoras' theorem,

$$\begin{aligned} x^2 + (2x - 1)^2 &= 1^2 \\ x^2 + 4x^2 - 4x + 1 &= 1 \\ 5x^2 - 4x &= 0 \\ x(5x - 4) &= 0 \end{aligned}$$

$x = 0$ gives a degenerate triangle and square, so the required solution is $x = \frac{4}{5}$.
Hence the area of the square is

$$\left(\frac{8}{5}\right)^2 = \frac{64}{25} = 2.56 \text{ units}^2$$

S2.



In the diagram ST is parallel to QR, UT is parallel to SR, $PU = 4$ cm and $US = 6$ cm. Find the length of SQ.

Solution

In triangle PSR we can say

$$\frac{4}{6} = \frac{PU}{US} = \frac{PT}{TR}.$$

Now in triangle PQR we can say

$$\frac{4}{6} = \frac{PT}{TR} = \frac{PS}{SQ} = \frac{10}{SQ}$$

Hence

$$SQ = 10 \times \frac{6}{4}.$$

So $SQ = 15$.

S3. Show that the equation of any circle passing through the points of intersection of the ellipse $(x + 2)^2 + 2y^2 = 18$ and the ellipse $9(x - 1)^2 + 16y^2 = 25$ can be written in the form

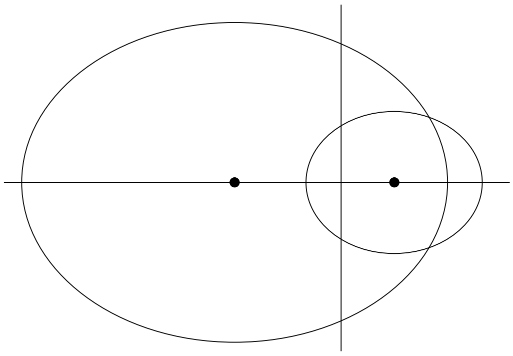
$$x^2 - 2ax + y^2 = 5 - 4a$$

where a is a constant.

Solution

The ellipse $(x + 2)^2 + 2y^2 = 18$ has centre $(-2,0)$ and passes through $\left(-2, \pm 3\right)$ and $\left(-2 \pm \sqrt{14}, 0\right)$.

The ellipse $9(x - 1)^2 + 16y^2 = 25$ has centre $(1, 0)$ and passes through $\left(1, \pm\frac{5}{4}\right)$, $\left(\frac{8}{3}, 0\right)$ and $\left(-\frac{2}{3}, 0\right)$.



A diagram shows that there are two crossing points.

Eliminating y^2:

$$9(x - 1)^2 - 8(x + 2)^2 = 25 - 8 \times 18$$

$$x^2 - 50x = -96$$

$$(x - 25)^2 - 625 = -96$$

$$(x - 25)^2 = 625 - 96 = 529 = 23^2$$

$$x = 25 - 23 = 2 \text{ or } x = 25 + 23 = 48$$

Alternatively, when $x = 48$, the y^2 values are negative, and so there is no real solution for y so the crossing points are at $x = 2$.

When $x = 2$, $(2 + 2)^2 + 2y^2 = 18$ and so $y = \pm 1$.

Any circle passing through both these points must have its centre on the x-axis, say at $(a,0)$. Thus its equation is

$$(x - a)^2 + y^2 = r^2$$

It also passes through the points $(2, \pm 1)$ and so

$$(2 - a)^2 + (\pm 1)^2 = r^2$$

$$r^2 = a^2 - 4a + 4 + 1 = a^2 - 4a + 5$$

So the equation is

$$(x - a)^2 + y^2 = a^2 - 4a + 5$$

$$x^2 - 2ax + a^2 + y^2 = a^2 - 4a + 5$$

$$x^2 - 2ax + y^2 = 5 - 4a.$$

as required.

S4. There are 6 pairs of siblings from 6 different families.

(a) How many ways can they be divided into *two* teams so that both siblings of a pair are never in the same team?

(b) How many ways can they be divided into *three* teams so that both siblings of a pair are never in the same team?

Solution

(a) There must be one sibling from each pair on the first team. There are 2 ways to choose a sibling from each pair, so 2^6 ways to choose the first team. But the teams are interchangeable, so divide by 2 to get 2^5 ways of choosing the two teams.

(b) Select the first team of four by first choosing the pairs A, B, C, D in $\frac{6 \times 5 \times 4 \times 3}{4!} = 15$ ways. Then choose one sibling from each of these pairs in 2 ways each time, giving $15 \times 2^4 = 240$ ways in all.

Then select the second team. It must include one each from pairs E and F (otherwise both siblings in these pairs would be in the third team), in $2^2 = 4$ ways. And also 2 of the remaining siblings from pairs A, B, C, D, in $\frac{4 \times 3}{2!} = 6$ ways. These can be combined to form the second team in $4 \times 6 = 24$ ways.

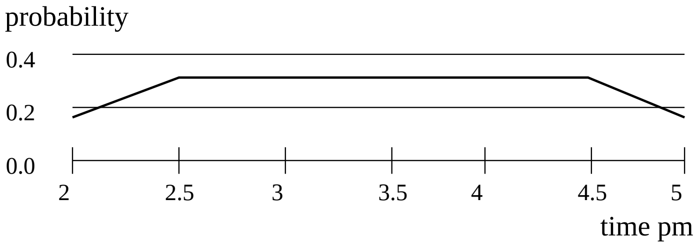
The remaining siblings are from different families and form the third team in 1 way.

Finally the three teams are interchangeable, so divide by $3! = 6$.

So there are $\frac{240 \times 24 \times 1}{6} = 960$ ways of selecting the three teams of four.

S5. Two forgetful friends agree to meet in a coffee shop one afternoon but each has forgotten the agreed time. Each remembers that the agreed time was somewhere between 2 pm and 5 pm. Each decides to go to the coffee shop at a random time between 2 pm and 5 pm, wait for half an hour, and leave if the other doesn't arrive. Find the probability that they meet.

Solution



time	probability
2	$\frac{1}{6}$
2.5	$\frac{1}{3}$
4.5	$\frac{1}{3}$
5	$\frac{1}{6}$

Call the friends A and B.

If A arrives between 2.30 pm and 4.30 pm (2.5 and 4.5 on the graph scale) then he will meet B provided B arrives within half an hour of A's arrival, either before or after. Then they will meet if B arrives in this one hour slot, out of his possible 3 hour range. Therefore the probability of meeting is 1/3.

If A arrives at 2 pm then he will meet B only if B arrives before 2.30 pm, giving a probability of (1/2)/3=1/6. If A arrives between 2 and 2.30 pm (2 and 2.5 on the graph scale) then they will meet provided B either arrives first or within half an hour after A. Thus the probability increases linearly between 2 and 2.30 pm from 1/6 to 1/3.

There is a similar effect when A arrives after 4.30 pm, as shown on the graph.

So the overall probability that they meet is

$$\frac{1}{6}\,\frac{\frac{1}{6}+\frac{1}{3}}{2} + \frac{2}{3} \times \frac{1}{3} + \frac{1}{6}\,\frac{\frac{1}{6}+\frac{1}{3}}{2} = \frac{11}{36}$$