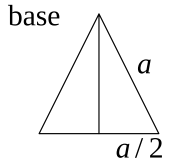
Senior Division 2016-2017 Round 1 Solutions

S1. A pyramid stands on horizontal ground. Its base is an equilateral triangle with sides of length a, the other three edges of the pyramid are of length b and its volume is V. Show that the volume of the pyramid is

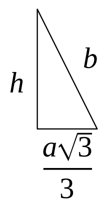
$$V = \frac{1}{12}a^2\sqrt{3b^2 - a^2}.$$

The pyramid is then placed so that a non-equilateral face lies on the ground. Find the height of the pyramid in this position.

Solution



vertical section through corner



Area of base of pyramid = $\frac{1}{2}a\sqrt{3}\,\frac{a}{2} = a^2\frac{\sqrt{3}}{4}$

The centre of the base of the pyramid is $\frac{2}{3}\sqrt{3}\,\frac{a}{2} = \frac{a\sqrt{3}}{3}$ from a corner.

Let the height of the pyramid be h.

$$h^2 = b^2 - \left(\frac{a\sqrt{3}}{3}\right)^2 = b^2 - \frac{a^2}{3}$$

$$h = \sqrt{\frac{3b^2 - a^2}{3}}$$

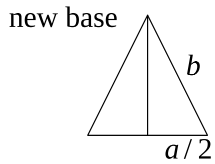
Hence

$$V = \frac{1}{3} \times \text{ area of base} \times \text{ height} = \frac{1}{3}a^2\frac{\sqrt{3}}{4}\sqrt{\frac{3b^2 - a^2}{3}}$$

$$= \frac{1}{12}a^2\sqrt{3b^2 - a^2}$$

When the pyramid is lying on a non-equilateral face: new height = $3V$/new base area and

$$\text{new base area} = \frac{1}{2}a\sqrt{b^2 - \frac{a^2}{4}} = \frac{1}{4}a\sqrt{4b^2 - a^2}$$



Hence

$$\text{new height} = \frac{3V}{\text{new base area}} = 3 \times \frac{1}{12}\frac{a^2\sqrt{3b^2 - a^2}}{\frac{1}{4}a\sqrt{4b^2 - a^2}}$$

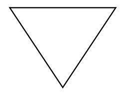
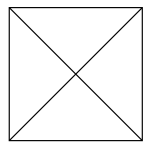
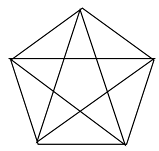
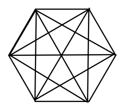
$$= \frac{a\sqrt{3b^2 - a^2}}{\sqrt{4b^2 - a^2}} = a\sqrt{\frac{3b^2 - a^2}{4b^2 - a^2}}$$

S2. Eight islands each have one or more air services. An air service consists of flights to and from another island, and no two services link the same pair of islands. There are 17 air services in all between the islands.

Show that it must be possible to use these air services to fly between any pair of islands.

Solution

If it is not possible then the air services divide the islands into two or more disconnected groups. The maximum number of services is when there are two groups − otherwise additional services could be added to reduce the number of groups to two. Every island has at least one service so the the smallest group size is 2, and the possible sizes of the two groups are 2, 6; 3, 5; and 4, 4. The maximum number of services within a group is when every island in the group is linked directly to every other island in the group:

group sizes	max. number of services within groups	
2	1	
3	3	
4	6	
5	10	
6	15	

group sizes		max. number of services within groups
2	6	1 + 15 = 16
3	5	3 + 10 = 13
4	4	6 + 6 = 12

Hence there can be at most 16 services when the islands are in two disconnected groups. So the 17th service must link the two groups and make it possible to fly between any pair of islands.

S3. How many distinct solutions consisting of positive integers does this system of equations have?

$$\begin{array}{ccccccccccccccccccc}
x_1 & + & x_2 & + & x_3 & & & & & & & & & & & & & = & 5 \\
& & & & & & y_1 & + & y_2 & + & y_3 & & & & & & & = & 5 \\
& & & & & & & & & & & & z_1 & + & z_2 & + & z_3 & = & 5 \\
x_1 & + & & & & & y_1 & + & & & & & z_1 & & & & & = & 5 \\
& & x_2 & + & & & & & y_2 & + & & & & & z_2 & & & = & 5 \\
& & & & x_3 & + & & & & & y_3 & + & & & & & z_3 & = & 5
\end{array}$$

Solution

Rearrange the variables into a square:

$$\begin{array}{lccccc}
 & & & & \text{total} \\
 & x_1 & x_2 & x_3 & 5 \\
 & y_1 & y_2 & y_3 & 5 \\
 & z_1 & z_2 & z_3 & 5 \\
\text{total} & 5 & 5 & 5 &
\end{array}$$

The only sets of three positive integers making 5 are {1,1,3} and {1,2,2}.

Case 1

Fix the first row as 1,1,3 and note that $y_3 = z_3 = 1$.

$$\begin{array}{lccccc}
 & & & & \text{total} \\
 & 1 & 1 & 3 & 5 \\
 & y_1 & y_2 & 1 & 5 \\
 & z_1 & z_2 & 1 & 5 \\
\text{total} & 5 & 5 & 5 &
\end{array}$$

Then y_1 could be 1, 2 or 3 and the remaining values follow.

There are 3 possible positions for the 3 in the first row and 3 possible solutions for the remaining square, making $3 \times 3 = 9$ in all for Case 1.

Case 2

Fix the first row as 1,2,2.

$$\begin{array}{lccccc}
 & & & & \text{total} \\
 & 1 & 2 & 2 & 5 \\
 & y_1 & y_2 & y_3 & 5 \\
 & z_1 & z_2 & z_3 & 5 \\
\text{total} & 5 & 5 & 5 &
\end{array}$$

Then y_3 could be 1 or 2.

Case 2a: take $y_3 = 1$ (and hence $z_3 = 2$).

$$\begin{array}{lccccc}
 & & & & \text{total} \\
 & 1 & 2 & 2 & 5 \\
 & y_1 & y_2 & 1 & 5 \\
 & z_1 & z_2 & 2 & 5 \\
\text{total} & 5 & 5 & 5 &
\end{array}$$

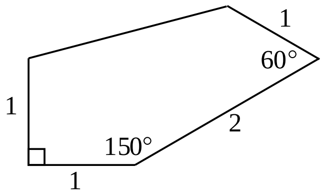
So z_1 could be 1 or 2 and then the remaining values follow.

Case 2b: take $y_3 = 2$ (and hence $z_3 = 1$). This time y_1 could be 1 or 2 to fix the remaining values.

Combining cases 2a and 2b, there are 3 possible positions for the 1 in the first row, 2 possible values for y_3 and 2 possible ways to fix the remaining square, making $3 \times 2 \times 2 = 12$ in all.

Adding case 1 and case 2 there are 21 possible positive integer solutions for the original equations.

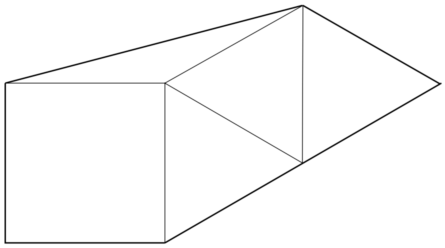
S4. Over many centuries, tilings have been important to mathematicians and to society. To use a single tiling to cover a plane is very useful. For a few cases, regular polygons (equilateral triangles, squares, regular hexagons) can be used but in order to do that, their angle needs to be a factor of 360°. It is not possible to have a tiling of regular pentagons. However, non-regular pentagons are known which will tile the plane. There was much excitement in August 2015 when a team in the United States discovered a hitherto unknown pentagonal tiling and it is shown in the diagram.



Use the lengths given and angle sizes to calculate its exact area.

Solution

The pentagon may be dissected as shown into a square, three equilateral triangles and an isosceles triangle.



The square has area 1.

Each equilateral triangle has area $\frac{1}{2} \times 1 \times 1 \times \sin 60° = \frac{\sqrt{3}}{4}$.

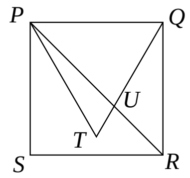
The equal sides of the isosceles triangle are of length 1 and the angle between them is $360° - 90° - 60° - 60° = 150°$ so its area is

$$\frac{1}{2} \times 1 \times 1 \times \sin 150° = \frac{1}{2} \times 1 \times 1 \times \sin 30° = \frac{1}{4}.$$

Hence the area of the pentagon is equal to

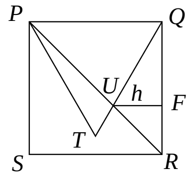
$$1 + \frac{3\sqrt{3}}{4} + \frac{1}{4} = \frac{5 + 3\sqrt{3}}{4}.$$

S5.



The diagram shows square $PQRS$ with sides of length 1 unit. Triangle PQT is equilateral. Show that the area of triangle UQR is $(\sqrt{3} - 1)/4$ square units.

Solution



Let the perpendicular from U to QR meet QR at F, and let UF have length h.
Then since $\angle URF = 45°$, $FR = UF = h$.

$$QF = QR - FR = 1 - h$$

$$\text{and } \angle UQF = 30°$$

$$\tan \angle UQF = \tan 30° = \frac{1}{\sqrt{3}} = \frac{h}{1 - h}$$

$$h\sqrt{3} = 1 - h$$

$$h = \frac{1}{1 + \sqrt{3}} = \frac{\sqrt{3} - 1}{2}$$

Hence area of triangle UQR is

$$\frac{1}{2} \times 1 \times \frac{\sqrt{3} - 1}{2} = \frac{\sqrt{3} - 1}{4}.$$