The Scottish Mathematical Council

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MATHEMATICAL CHALLENGE 2016-2017

Entries must be the unaided efforts of individual pupils.
Solutions must include explanations and answers without explanation will be given no credit.
Do not feel that you must hand in answers to all the questions.

CURRENT AND RECENT SPONSORS OF MATHEMATICAL CHALLENGE ARE
The Edinburgh Mathematical Society, The Maxwell Foundation, Professor L E Fraenkel, The London Mathematical Society and The Scottish International Education Trust.

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Senior Division: Problems 1

S1. A pyramid stands on horizontal ground. Its base is an equilateral triangle with sides of length a, the other three edges of the pyramid are of length b and its volume is V. Show that the volume of the pyramid is

$$V = \frac{1}{12}a^2\sqrt{3b^2 - a^2}.$$

The pyramid is then placed so that a non-equilateral face lies on the ground. Find the height of the pyramid in this position.

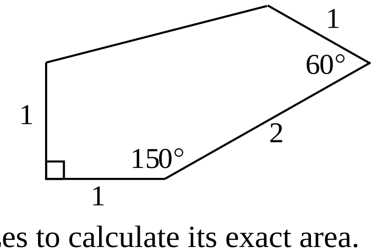
S2. Eight islands each have one or more air services. An air service consists of flights to and from another island, and no two services link the same pair of islands. There are 17 air services in all between the islands.

Show that it must be possible to use these air services to fly between any pair of islands.

S3. How many distinct solutions consisting of positive integers does this system of equations have?

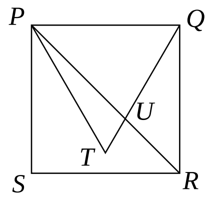
$$\begin{array}{ccccccccccccccccc}
x_1 & + & x_2 & + & x_3 & & & & & & & & & & & = & 5 \\
 & & & & & & y_1 & + & y_2 & + & y_3 & & & & & = & 5 \\
 & & & & & & & & & & & & z_1 & + & z_2 + z_3 & = & 5 \\
x_1 & + & & & & & y_1 & + & & & & & z_1 & & & = & 5 \\
 & & x_2 & + & & & & & y_2 & + & & & & & z_2 & = & 5 \\
 & & & & x_3 & + & & & & & y_3 & + & & & z_3 & = & 5
\end{array}$$

S4. Over many centuries, tilings have been important to mathematicians and to society. To use a single tiling to cover a plane is very useful. For a few cases, regular polygons (equilateral triangles, squares, regular hexagons) can be used but in order to do that, their angle needs to be a factor of 360°. It is not possible to have a tiling of regular pentagons. However, non-regular pentagons are known which will tile the plane. There was much excitement in August 2015 when a team in the United States discovered a hitherto unknown pentagonal tiling and it is shown in the diagram.



Use the lengths given and angle sizes to calculate its exact area.

S5.



The diagram shows square *PQRS* with sides of length 1 unit. Triangle *PQT* is equilateral. Show that the area of triangle *UQR* is $(\sqrt{3} - 1)/4$ square units.

END OF PROBLEM SET 1