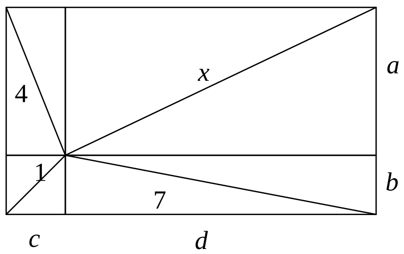
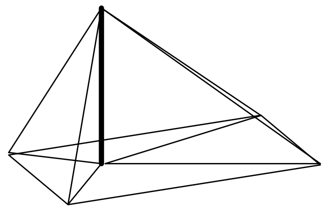
# 2015-2016 Senior Solutions Round 2

**S1** A radio ham places an aerial mast where it gives the best reception on the roof of his rectangular garage. He then fixes wire supports from the top of the mast to each corner of the roof. The lengths of two opposite supports are seven metres and four metres and the length of one of the others is 1 metre. Find the length of the remaining support.

*Solutions*



Let the length of the remaining wire support be $x$ metres.
Let the height of the mast be $h$ metres and the distances of the base of the mast from the sides of the roof be as shown in the diagram above.
Then

$$a^2 + c^2 + h^2 = 4^2$$
$$b^2 + c^2 + h^2 = 1^2$$
$$b^2 + d^2 + h^2 = 7^2$$

Hence

$$(a^2 + c^2 + h^2) + (b^2 + d^2 + h^2) - (b^2 + c^2 + h^2) = 16 + 49 - 1 = 64$$

The left-hand side reduces to $a^2 + d^2 + h^2$ but this is also the expression for $x^2$. So

$$x^2 = 64.$$

Hence $x = 8$ so the length of the remaining wire support is 8 metres.

**S2** A large board has 1000 + signs and 999 − signs written on it. Any two symbols can be deleted provided they are replaced as follows:

- if the deleted symbols are the same, they are replaced by a +
- if the deleted symbols are different they are replaced by a −.

Repeat this process until there is only one symbol left. Which symbol is it and why?

*Solutions*

If two + signs are replaced by one +, the number of − signs remains unchanged.

If two − signs are replaced by one +, the number of − signs reduces by 2.

If a + sign and a − sign are replaced by one −, the number of − signs remains unchanged.

So in whatever order the signs are deleted, the number of − signs must always be odd since we started with 999, an odd number.

So when there is only one sign left, it must be a − sign to make the number of − signs odd.

(Whilst this solution is short, it is not easy to find!)

**S3.** There are five beads on a metal ring, each with a number on. When the beads are numbered 1,2,3,4,5 consecutively round the ring, show that it is possible to write every whole number from 1 to 15 as the total of the numbers on adjacent beads.

Now assume that you are allowed to number the beads yourself. Each bead may be labelled with *any* positive whole number, but the numbers do not have to be in sequence and they do not have to form a set of five consecutive numbers. What is the highest number *N* for which you can number the beads in such a way that you can write every whole number from 1 to *N* as the total of the numbers on adjacent beads?

*Solutions*

Totals 1,2,3,4,5 are obtained using single beads. 6 is 5 + 1. 7 is 4 + 3. The totals from 8 to 14 are the beads left behind when those required for totals 1 to 7 are selected. 15 is all the beads.

There are 5 ways of selecting one bead.

There are 5 ways of selecting two adjacent beads.

There are 5 ways of selecting three adjacent beads.

There are 5 ways of selecting four adjacent beads.

There is one way of selecting all five beads.

Hence there are at most 21 different totals, and so the maximum possible total of all five beads is 21.

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To obtain the total 1 there must be a bead numbered 1. There cannot be a second bead numbered 1, as then the total 1 could be obtained in 2 ways. Hence to obtain the total 2 there must be a bead numbered 2.

(i) If beads 1 and 2 are separated, then to obtain the total 3 there must be bead numbered 3.

If bead 3 is placed between bead 1 and bead 2, then all totals up to 6 can be made, and so the other two beads must be 7 and 8 to make the overall total 21. But this doesn't work: if bead 7 is next to bead 1 then total 8 can be obtained in two ways, and if bead 7 is next to bead 2 then total 9 can be obtained as 7 + 2 or 8 + 1.

Try bead 3 next to bead 1 but not bead 2. This supplies total 4, so next a bead numbered 5 is required, and also a bead numbered 10 to make the overall total 21. If these are arranged

1 3 5 2 10

then total 6 cannot be obtained.

Try the arrangement

3 1 5 2 10

Each single bead supplies a total. 4 is 1 + 3. 6 is 5 + 1. 7 is 5 + 2. 8 is 1 + 5 + 2. 9 is 1 + 3 + 5. The totals from 11 to 20 are the adjacent beads left when the totals 1 to 10 are selected. Finally 21 is the total of all 5 beads.

This solves the problem: we have shown that 21 is the maximum possible total and that it is actually obtainable by finding an example solution.

However (for a bonus mark) we can continue and show that this example solution is the only solution:

(ii) If beads 1 and 2 are adjacent, the total 3 is 1 + 2. So the next bead is numbered 4, and the remaining two beads total 21-7 = 14: either 5 and 9 or 6 and 8. If bead 4 is placed next to bead 1, then their total is 5, and so the remaining two beads must be numbered 6 and 8. But neither arrangement 4 1 2 8 6 (no 9) nor 4 1 2 6 8 (two 8s) work. If bead 4 is placed next to bead 2, then their total is 6, and hence the remaining two beads must be numbered 5 and 9. But neither arrangement 1 2 4 5 9 (no 8) nor 1 2 4 9 5 (no 10) work. If bead 4 is placed opposite beads 1 and 2 then the possible arrangements are 1 2 5 4 9 (no 6), 1 2 9 4 5 (no 7), 1 2 6 4 8 (two 8s) and 1 2 8 4 6 (no 5). Again none of these work.

So the only possible arrangement is 3 1 5 2 10.

**S4.** I have five regular tetrahedra that I use as dice. On each face of each tetrahedron is one of the numbers 1, 2, 3 or 4 (where numbers may be repeated on faces of a single tetrahedron). No two tetrahedra use the same set of four numbers, and for each tetrahedron the sum of the four numbers is 10.

I play a game with my friend using four of these dice. He picks up a die, then I pick up a die, and we both throw. Whoever has the higher face-down number wins, or it may be a draw.

You would expect that this game to be fair, since the total on each of the five dice is 10. However by carefully selecting which one of the five dice to leave out before we start to play, I can ensure that, whichever die my friend chooses, I have a better chance of winning than he does. Which die must I leave out?

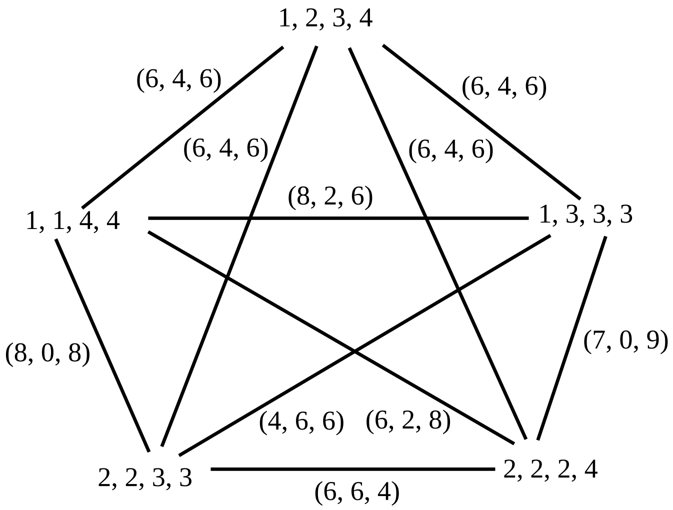
We then play the game with the remaining four dice (after leaving out the die selected in the previous paragraph). If my friend chooses the die with faces numbered 2, 2, 2, 4, which die will give me the highest probability of winning, and with what probability?

*Solutions*

The possible dice are

1, 1, 4, 4; 1, 2, 3, 4; 1, 3, 3, 3; 2, 2, 2, 4; 2, 2, 3, 3.

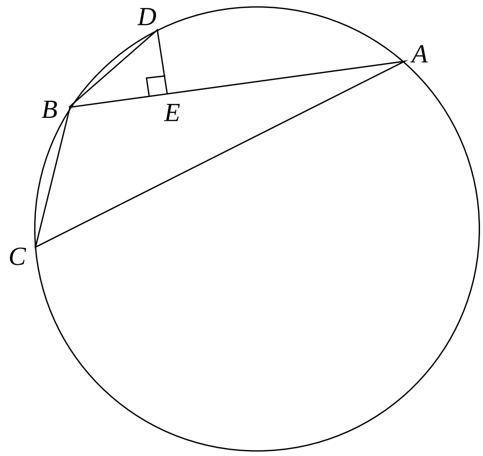
Placing the five dice at the corners of a pentagon, the triple (win left, draw, win right) shows the number of ways that firstly the leftmost die at the line end can win, secondly there is a draw, and thirdly the die at the rightmost line end can win. (The total is always 16.)




The fair die is 1, 2, 3, 4 which I should remove to leave myself with the advantage.

When my friend chooses the die 2, 2, 2, 4, I should choose die 1, 3, 3, 3, giving me a probability winning of 9/16.

**S5.** In the figure, $D$ is the midpoint of the arc $AC$ of a circle. The point $B$ is on the arc between $C$ and $D$, and $E$ is the foot of the perpendicular from $D$ to $AB$. Show that $E$ is the midpoint of the path from $A$ to $C$ along the line segments $AB$ and $BC$.




*Alternative Solution*

Join $D$ to $A$ and to $C$. As we know that $D$ is the midpoint of the arc $AC$ it follows that $DC = DA$.

Now extend the chord $CB$ and draw the perpendicular to it from $D$. Call this point $E'$.

We now show that $BE = BE'$ by proving that triangles $BED$ and $BE'D$ are congruent. We have

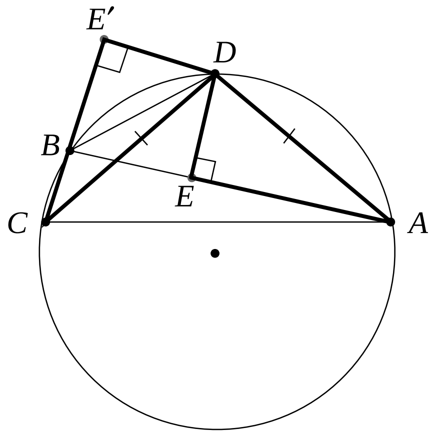
$$\angle DBE' = \angle DAC \qquad \{\text{angles in alternate segments}\}$$

$$\angle DAC = \angle DCA \qquad \{\triangle ADC \text{ is isosceles}\}$$

$$\angle DCA = \angle DBA \qquad \{\text{both are subtended by the chord } DA\}$$

So as $\angle DBA$ is the same angle as $\angle DBE$ we have $\angle DBE' = \angle DBE$. As $\angle BED = \angle BE'D\,(= 90°)$, triangles $BED$ and $BE'D$ are equiangular and as $BD$ is common they are congruent.

Thus $BE = BE'$.




We now note that $\angle BCD$ and $\angle BAD$ are subtended by the chord $BD$ so they are equal. So as these angles are equal and $\angle AED = \angle CE'D\,(= 90°)$ and $DA = DC$ we have proved that triangles $AED$ and $CE'D$ are congruent and hence $AE = CE'$. Therefore $CB + BE = CB + BE' = CE' = EA$. So $E$ is the midpoint of the path from $A$ to $C$ along the line segments $AB$ and $BC$.