2015-2016 Senior Solutions Round 1

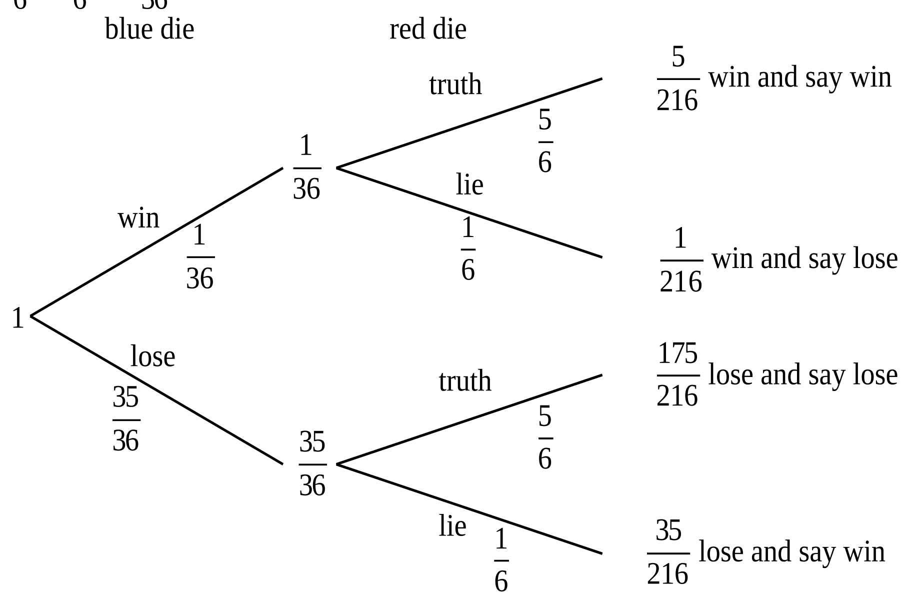
S1. I have two blue dice and one red die.
I use the blue dice to play a simple game: if I roll a double six, I win. Otherwise, I lose.
I also roll the red die. If I roll a one, I'll lie about whether I've won or lost the game; if I roll any other number, I'll tell the truth.

I roll all three dice.
I turn to you and say "I won!".

What is the probability that I did in fact win the game?

Solution

P(win) = $\frac{1}{6} \times \frac{1}{6} = \frac{1}{36}$ (A)



$$\frac{5 + 1 + 175 + 35}{216} = 1$$

P(say win) = P(win and say win) + P(lose and say win)

$$= \frac{5}{216} + \frac{35}{216} = \frac{40}{216} \qquad \text{(B)}$$

P(win given say win) = P(win and say win)/P(say win)

$$= \frac{\frac{5}{216}}{\frac{40}{216}} = \frac{5}{40} = \frac{1}{8}$$

So when I say I won the game, the probability that I did in fact win is $\frac{1}{8}$.

S2. A coach travels over a hilly route from town A in the highlands to town B by the coast. Going uphill it travels at 42 mph, going downhill it travels at 56 mph and on level ground it travels at 48 mph. It takes 2 hours and 20 minutes to travel from A to B and 2 hours and 40 minutes to travel back. Find the distance between A and B.

Solution

Let the distance from A to B be u miles up, d miles down and f miles on the flat.

Then

$$\frac{u}{42} + \frac{d}{56} + \frac{f}{48} = \frac{7}{3}$$

and on the return route where uphill and downhill exchange

$$\frac{d}{42} + \frac{u}{56} + \frac{f}{48} = \frac{8}{3}.$$

Adding:

$$(d + u)\left(\frac{1}{42} + \frac{1}{56}\right) + \frac{2f}{48} = 5$$

$$(d + u) \times \frac{2}{48} + \frac{2f}{48} = 5$$

$$d + u + f = \frac{5 \times 48}{2} = 120.$$

Hence the distance is 120 miles.

Now check that this is possible, since u could work out to be negative!

Subtracting:

$$\frac{d - u}{42} - \frac{d - u}{56} = \frac{1}{3}$$

$$(d - u)\left(\frac{1}{42} - \frac{1}{56}\right) = \frac{1}{3}$$

$$d - u = 56$$

If $u = 0$, then $d = 56$ and $f = 64$.

The hilly route suggests that $u > 0$, but that is certainly possible:

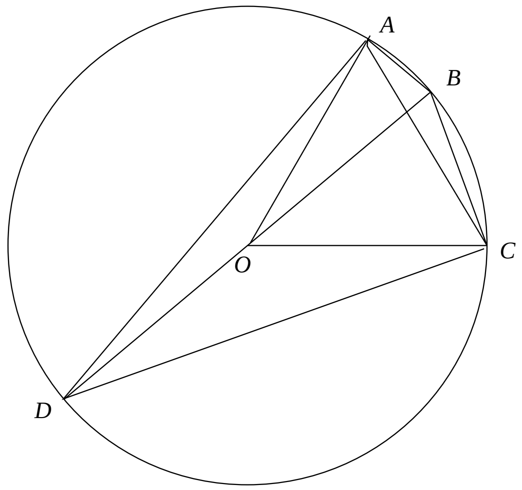
e.g. if $u = 28$, then $d = 84$ and $f = 8$

i.e. 28 miles uphill, 84 miles downhill and 8 miles on the flat.

S3. A convex polygon with 12 sides is inscribed in a circle. This polygon has six sides of length $\sqrt{2}$ and six sides of length $\sqrt{24}$ in some order. What is the radius of the circle?

Solution

There must be at least one side of length $\sqrt{2}$ which is adjacent to a side of length $\sqrt{24}$.



Let these sides be AB and BC respectively and let the centre of the circle be O. The arc from A to C is 1/6 of the total circumference of the circle. So the angle AOC is 60°. Since $AO = CO$ is the radius r of the circle, AC is also equal to r. So the triangle ABC has sides of length $\sqrt{2}$, $\sqrt{24}$ and r.

Draw the diameter BOD.

Then $\angle AOC$ is twice $\angle ADC$ as they are subtended by the same chord AC so $\angle ADC = \frac{1}{2}\angle AOC = 30°$.

But since $ABCD$ is cyclic, $\angle ABC + \angle ADC = 180°$. So $\angle ABC = 150°$.

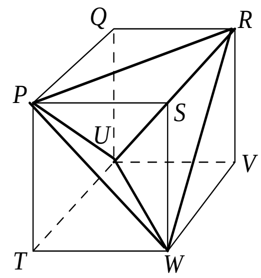
Now applying the cosine rule to triangle ABC gives

$$r^2 = \left(\sqrt{2}\right)^2 + \left(\sqrt{24}\right)^2 - 2\sqrt{2}\sqrt{24}\left(\frac{-\sqrt{3}}{2}\right) = 38.$$

So the radius is $\sqrt{38}$.

S4. The diagram shows a tetrahedron $PRWU$ which fits snugly inside a cube $PQRSTUVW$.

Find the ratio of the surface area of the cube to the surface area of the tetrahedron .



Solution

Let the cube have side a.

Then the surface area of each face is a^2, and the total surface area is $6a^2$.

Each face of the tetrahedron is an equilateral triangle with side $\sqrt{2}a$. The height of these triangles is $\sqrt{3/2}a$ and the area is $\sqrt{3}a^2/2$. So the four faces have total area $2\sqrt{3}a^2$.

So the ratio is $6a^2 : 2\sqrt{3}a^2$ or $\sqrt{3} : 1$.

S5. Let n be a three-digit number and let m be the number obtained by reversing the order of the digits in n. Suppose that m does not equal n and that $n + m$ and $n - m$ are both divisible by 7. Find all such pairs n and m.

Solution

Since $(n + m) + (n - m) = 2n$ and $(n + m) - (n - m) = 2m$, 7 divides both $2n$ and $2m$; hence 7 divides both n and m.

Let $n = 100a + 10b + c$; then $m = 100c + 10b + a$.

We can assume, by interchanging n and m if necessary, that $a > c$ (noting that $a \neq c$). Since we can express 100 as $7 \times 14 + 2$ and 10 as $7 + 3$, we have

$$n = (14 \times 7 + 2)a + (7 + 3)b + c,$$

and, since n is divisible by 7, so is $2a + 3b + c$. Similarly, $2c + 3b + a$ is also divisible by 7. Subtracting these,

$$(2a + 3b + c) - (2c + 3b + a) = a - c,$$

so we deduce that $a - c$ is divisible by 7. Since a and c are integers between 1 and 9 and, by assumption, $a > c$ and $a - c = 7$. There are only two possibilities:

$$\text{either } a = 8 \text{ and } c = 1; \text{ or } a = 9 \text{ and } c = 2.$$

Since $2a + 3b + c$ is divisible by 7, the first case gives $3b + 17$ is divisible by 7 and hence $3b + 3 = 3(b + 1)$ is as well, giving $b = 6$. In the second case, $3b + 20$ is divisible by 7 and hence $3b - 1$ is as well, giving $b = 5$. Thus the only possible pairs of numbers are $\{861, 168\}$ and $\{952, 259\}$.