2014-2015 Senior Solutions Round 2

S1

A fruit drink manufacturer has a mixture of 100 litres containing w% of pure orange juice. By adding x litres of a mixture containing y% of pure orange juice he wishes to produce a mixture containing z% of pure orange juice. Find the value of x in terms of w, y and z.

Solution

	litres of mixture	% pure orange juice	litres of pure orange juice
start	100	w	w
amount added	x	y	$\frac{xy}{100}$
total	$100 + x$	z	$w + \frac{xy}{100}$

The number of litres of pure orange juice in the final mixture is also

$$\frac{(100 + x)z}{100}$$

Hence

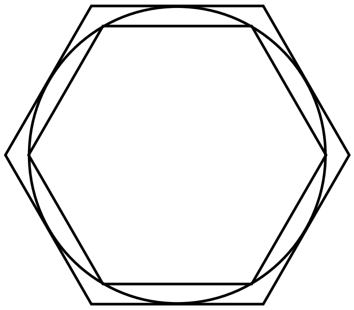
$$\frac{(100 + x)z}{100} = w + \frac{xy}{100}$$

$$(100 + x)z = 100w + xy$$

$$(z - y)x = 100w - 100z = 100(w - z)$$

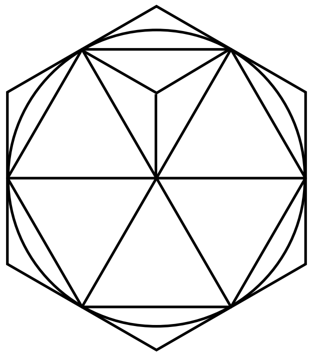
$$x = \frac{100(w - z)}{z - y}$$

S2



A regular hexagon circumscribes a circle which circumscribes another regular hexagon. The inner hexagon has an area of 3 square units. What is the area of the outer hexagon?

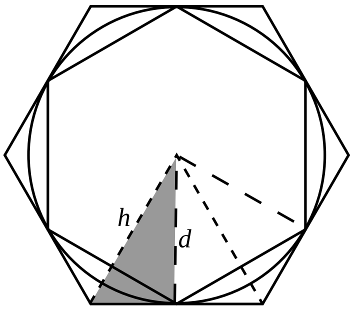
Solution



First, rotate the outer hexagon so that the vertices of the inner hexagon touch the outer hexagon. The radial line of the circle to a vertex of the inner hexagon is then at right angles to the edge of the outer hexagon. If one of the equilateral triangles of the inner hexagon is split into three isosceles triangles, as shown, the angle of each of these isosceles triangles at a vertex of the hexagon is 30 degrees. Hence the angle at that vertex of the triangle outside the inner hexagon is also 30 degrees. So the area of the triangle outside the inner hexagon is equal to the area of one of these isosceles triangles. So the area of the outer hexagon is 4/3 times the area of the inner hexagon i.e. is 4 square units.

(This can, of course, also be done without the initial rotation).

Alternative



Each hexagon can be split into six equilateral triangles. Let the sides be h and d as indicated.

The area of each of the larger triangles is $\frac{1}{2}h^2 \sin 60°$ and the area of each of the smaller triangles is $\frac{1}{2}d^2 \sin 60°$. Thus

$$\frac{\text{Area of larger hexagon}}{\text{Area of smaller hexagon}} = \frac{6 \times \left(\frac{1}{2}h^2 \sin 60°\right)}{6 \times \left(\frac{1}{2}d^2 \sin 60°\right)} = \frac{h^2}{d^2}.$$

But, from the shaded right-angled triangle

$$\frac{d}{h} = \sin 60° = \frac{\sqrt{3}}{2} \Rightarrow \frac{h}{d} = \frac{2}{\sqrt{3}} \Rightarrow \frac{h^2}{d^2} = \frac{4}{3}.$$

Thus

$$\text{Area of larger hexagon} = \frac{4}{3} \times \text{Area of smaller hexagon} = \frac{4}{3} \times 3 = 4.$$

S3

Four pirates were shipwrecked on a remote island. When they explored the island they found a cache of gold coins and agreed to share their haul equally. The night before they were due to divide it up, one pirate, worried that he would not get his fair share, secretly noticed that if he removed one coin, then the remainder would be exactly divisible by 4. So he took a coin and took a quarter of the remainder. Soon after, a second pirate decided to make sure of his share. Again, he found that, if he took one coin from what was left by the first pirate, the remainder would be divisible by 4. So he took a coin and took a quarter of what was left. The same thing was done in turn by the third and fourth pirates.

When they gathered the next morning to divide up the cache, with each pirate looking as innocent as possible, they found that the number of coins left was divisible by 4. What is the smallest number of coins in the original cache for this to be possible?

Solution

Let n be the original number of coins, so that the first pirate leaves $\frac{3}{4}(n - 1)$ coins. The second pirate will then leave

$$\frac{3}{4}\left\{\frac{3(n-1)}{4} - 1\right\} = \frac{3(3n-7)}{16};$$

the third pirate will leave

$$\frac{3}{4}\left\{\frac{3(3n-7)}{16} - 1\right\} = \frac{3(9n-37)}{64};$$

and then the fourth will leave

$$\frac{3}{4}\left\{\frac{3(9n-37)}{64} - 1\right\} = \frac{3(27n-175)}{256}$$

coins. Since this is divisible by 4, we can write

$$\frac{3(27n-175)}{256} = 4k$$

for some positive integer k, giving $3(27n - 175) = 4 \times 256k = 4^5k$. From this, we see that k is divisible by 3, say $k = 3m$, and then $27n = 175 + 4^5m = 175 + 1024m$. We now have

$$27n = (6 \times 27 + 13) + (38 \times 27 - 2)m$$

and hence $13 - 2m$ is divisible by 27. The smallest positive integer m for which this true is 20, giving

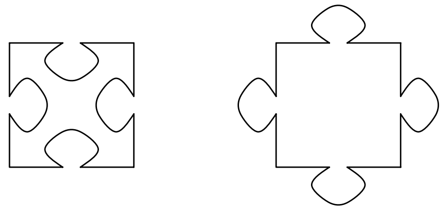
$$n = \frac{175 + 1024 \times 20}{27} = 765$$

as the smallest possible number of coins in the original cache.

As a check, starting with 765 coins, the first pirate leaves $\frac{3}{4} \times 764 = 573$ coins; the second leaves $\frac{3}{4} \times 572 = 429$; the third leaves $\frac{3}{4} \times 428 = 321$; and the fourth leaves $\frac{3}{4} \times 320 = 240$, which is indeed divisible by 4.

S4

A simple jigsaw puzzle has 225 pieces and is made from a 15×15 board where each square has area 1. All the internal (i.e. non-edge) pieces are of one of the two types shown and these types alternate along each row and column. If the ratio of the area of all the edge-pieces together to the area of the whole puzzle is $\frac{169}{675}$, how many pieces of each type are there? What is the ratio of the area of the two pieces shown?



Solution

Let us call the bits cut out of the square or added to the square as shown above "blobs". So the left hand one which we call type A, is formed by removing 4 blobs from the square and the right hand one, type B, has 4 blobs added to the square.

However the edge-pieces are joined to each other, each completed edge will consist of 15 squares with either 7 blobs added and 6 cut out or 7 blobs cut out and 6 added. The first case will occur if the corner piece of the internal square has type A and the second case will occur if that corner piece is of type B. In the first case the area of the edge will be 15 squares plus the area of one blob. Since this is repeated round the 4 edges the total area of the edge pieces in the first case will be 56 plus area of 4 blobs. In the second case, the area of the edge-pieces will be 56 minus the area of 4 blobs. Now

$$\frac{56}{225} = \frac{168}{675}$$

and so we are told the area of the edge pieces is greater than 56 so must be the first case. Thus there is one more piece of type A than there is of type B i.e. 85 pieces of type A and 84 of type B.

Also $4 \times \frac{\text{area of a blob}}{225} = \frac{1}{675}$ so the area of 4 blobs is $\frac{1}{3}$.

So type A has area $1 - \frac{1}{3} = \frac{2}{3}$ and type B has area $1 + \frac{1}{3} = \frac{4}{3}$.

So ratio of the areas $A : B$ is 1 : 2.

S5

The court mathematician once received his salary for a year's services all at one time and all in silver pound coins. He proceeded to arrange his coins in nine unequal piles so that in a 3×3 array, the number in each pile formed a magic square. The king looked and admired but complained that there was not a single prime number in this magic square. "If I had but nine coins more," said the mathematician, "I could add one coin to each pile and make a magic square with every number prime". The king was about to give him nine more coins when the court jester said "Wait". Then he removed one coin from each pile and this also gave a magic square with every number prime. What is the smallest number of coins that the mathematician could have received for his year's salary?

Solution

In 3×3 a magic square, three times the middle entry M is the common sum of each row, column and diagonal. The mathematician's total salary would be $9M$ so we need to find the smallest value of M which satisfies all the conditions.

Each entry X is such that both $X - 1$ and $X + 1$ are prime numbers. That means that X must be divisible by both 2 and 3. So all the entries are of the form $6N$ where $6N - 1$ and $6N + 1$ are both primes. We can further reduce the search for such numbers by noting that $6N$ cannot be of the form $5Y + 1$ or $5Y - 1$ otherwise 5 would divide $6N - 1$ or $6N + 1$ and so these would not be prime unless $6N = 6$. Thus the numbers we have to consider are 6 and those of the form $30N$, $30N + 12$, $30N + 18$.

Note that the middle entry M is the average of the outside two in each row, column and diagonal. Thus, one of the numbers which is M, must be the average of two others in 4 different ways. This dictates how long a list of candidates we need to consider. The following numbers are possible entries for the magic square (their deduction is a bit tedious- but easily available from the internet): 6, 12, 18, 30, 42, 60, 72, 102, 108, 138, 150, 180, 192, 198, 228, 240, 270, 282, 312, ….

Now list these to see which is the smallest one that can be written as the average of two others in at least four different ways.

Number	Double	Pairs					
42	84	12 + 72					
60	120	12+108	18+102				
72	144	6 + 138	42+102				
102	204	6 + 198	12+192				
108	216	18 + 198.					
138	276	6 + 270.					
150	300	18 + 282	30+270	60+240	72+228	102+198	108+192

Thus we have a candidate here as $M = 150$. We need to show that some of these numbers can be used to form a magic square. A little bit of trial gives:

60	282	108
198	150	102
192	18	240

Thus the smallest the mathematician's salary can be is $9 \times 150 = 1350$.

(Note that adding one or subtracting one to each entry of a magic square gives a magic square.)