2014-2015 Senior Round 1 solutions

S1

Two ferry boats set out at the same time from opposite banks of a loch. One boat is faster than the other and they pass each other at a point 650 metres from the nearer bank. After arriving at their destinations, each boat remains for 15 minutes to change passengers and then sets out on the return journey. This time, they meet at a point 350 metres from the other bank. How wide was the loch?

Solution

Since each boat spends the same time at rest, we can ignore that time and assume that they simply turn round and sail back immediately.

Let the width of the river be W metres and let the speeds of the boats be u and v in metres per second respectively, where we assume $u \leqslant v$.

Then, on the first crossing, the time taken to meet is

$$\frac{650}{u} = \frac{W - 650}{v} \Rightarrow \frac{u}{v} = \frac{650}{W - 650}$$

Similarly, on the second crossing time taken to meet is

$$\frac{W + 350}{u} = \frac{W + W - 350}{v} \Rightarrow \frac{u}{v} = \frac{W + 350}{2W - 350}$$

Hence

$$\begin{aligned} \frac{650}{W - 650} &= \frac{W + 350}{2W - 350} \\ 650(2W - 350) &= (W + 350)(W - 650) \\ 1300W - 650 \times 350 &= W^2 - 300W - 350 \times 650 \\ 1600W &= W^2 \end{aligned}$$

which, as $W \neq 0$, gives $W = 1600$ so the width of the loch is 1600 metres

S2

Given that $x^2 + \frac{1}{x^2} = 7$ and $x > 0$, find the exact value of $x^5 + \frac{1}{x^5}$ in its simplest form.

Solution

$$\left(x + \frac{1}{x}\right)^2 = x^2 + \frac{1}{x^2} + 2 = 7 + 2 = 9$$

so

$$x + \frac{1}{x} = 3$$

(taking the positive square root because $x > 0$).

$$\left(x^2 + \frac{1}{x^2}\right)\left(x + \frac{1}{x}\right) = x^3 + \frac{1}{x^3} + x + \frac{1}{x}$$

so

$$x^3 + \frac{1}{x^3} = 7 \times 3 - 3 = 18$$

$$\left(x^2 + \frac{1}{x^2}\right)\left(x^3 + \frac{1}{x^3}\right) = x^5 + \frac{1}{x^5} + x + \frac{1}{x}$$

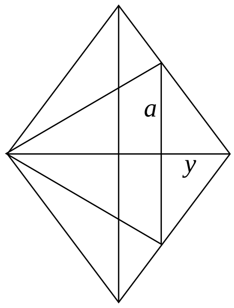
so

$$x^5 + \frac{1}{x^5} = 7 \times 18 - 3 = 126 - 3 = 123.$$

S3

The lengths of the diagonals of a rhombus are 6 and 8. An equilateral triangle inscribed in this rhombus has one vertex at an end-point of the shorter diagonal and one side parallel to the longer diagonal. Determine the length of a side of this triangle.
Express your answer in the form $k(4\sqrt{3} - 3)$ where k is a vulgar fraction.

Solution



Let the side of the equilateral triangle be $2a$. Then its altitude is $\sqrt{3}a$ and

$$y = \text{length of shorter diagonal} - \text{altitude}$$
$$= 6 - \sqrt{3}a$$

Also, by similar triangles,

$$a/y = 4/3$$
i.e. $4y = 3a$

Hence, as $y = 6 - \sqrt{3}a$,

$$4(6 - \sqrt{3}a) = 3a$$
$$(3 + 4\sqrt{3})a = 24$$

Multiplying both sides by $(4\sqrt{3} - 3)$:

$$39a = 24(4\sqrt{3} - 3)$$
$$2a = \frac{48}{39}(4\sqrt{3} - 3)$$

i.e. the sides of the equilateral triangle are $\frac{16}{13}(4\sqrt{3} - 3)$,

S4

In the following multiplication, each letter represents a different digit between 1 and 9.

$$\begin{array}{cccc} & A & B & C \\ \times & & D & E \\ \hline & F & E & C \\ D & E & C & \\ \hline H & G & B & C \end{array}$$

What two numbers are being multiplied together?

Solution

$E \times ABC = FEC$ and so $E \times C$ has unit digit C. Then $E \times C - C$ has unit digit 0; hence it is divisible by 10. So $(E - 1) \times C$ is divisible by 10 which is 2×5. It follows that either $E - 1$ is even and $C = 5$ or $E - 1 = 5$ and C is even.

Similarly, $D \times ABC = DEC$ and this implies that either $D - 1$ is even and $C = 5$ or $D - 1 = 5$ and C is even. Since D and E are different, we have $C = 5$ and both D and E are odd.

The multiplication sum now has the form

$$\begin{array}{cccc} & A & B & 5 \\ & & D & E \\ \hline & F & E & 5 \\ D & E & 5 & \\ \hline H & G & B & 5 \end{array}$$

and so $E + 5$ has unit digit B. Thus $E + 5 - B$ is a multiple of 10. Remembering that E and B are between 1 and 9, this means that $E + 5 - B$ must equal 0 or 10. Hence either $B = E + 5$ or $E = B + 5$. Since E is odd, $B = E + 5$ gives only two possibilities:

(a) $E = 1, B = 6$ or $E = 3, B = 8$,

whilst $E = B + 5$ gives

(b) $E = 7, B = 2$ or $E = 9, B = 4$.

Now consider $E \times AB5 = FE5$. The tens digit of $E \times 5$ is $\frac{1}{2}(E - 1)$ and so $E \times B + \frac{1}{2}(E - 1) - E$ has units digit 0. Checking the possibilities (a) and (b) above, we must have $E = 7$ and $B = 2$.

Finally, $D \times AB5 = DE5$ gives $D \times A25 = D75$ and the only possibility for this is when $D = 3$ and $A = 1$. So the multiplication sum is the working for

$$125 \times 37.$$

Writing it out in detail, we have

$$\begin{array}{cccc} & 1 & 2 & 5 \\ & & 3 & 7 \\ \hline & 8 & 7 & 5 \\ 3 & 7 & 5 & \\ \hline 4 & 6 & 2 & 5 \end{array}$$

S5

A *multiply-perfect number* is one for which the sum of its distinct factors, including 1 and the number itself, is an integer multiple of the given number. Show that 30240 is a multiply-perfect number.

Solution

$$30240 = 2^5 \times 3^3 \times 5 \times 7$$

factors					total
powers of 3	1	3	9	27	40
powers of 3 and 5	5	15	45	135	5×40
powers of 3 and 7	7	21	63	189	7×40
powers of 3 and 5 × 7	35	105	315	945	35×40
total					48×40

bringing in powers of 2	$1 \times 48 \times 40$
	$2 \times 48 \times 40$
	$4 \times 48 \times 40$
	$8 \times 48 \times 40$
	$16 \times 48 \times 40$
	$32 \times 48 \times 40$
total	$63 \times 48 \times 40$

So the overall total of factors is $63 \times 48 \times 40$
and $63 \times 48 \times 40 = (3^2 \times 7) \times (2^4 \times 3) \times (2^3 \times 5) = 2^7 \times 3^3 \times 5 \times 7 = 4 \times 30240$.
i.e. the overall total of factors is 4 times the original 30240. So 30240 is indeed a multiply-perfect number.