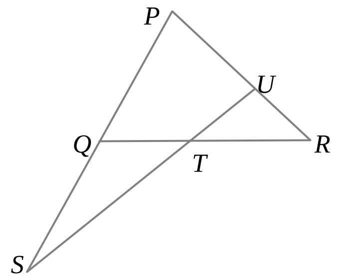
2013-2014 Senior Solutions Round 2

S1.

PQR is any triangle. The side *PQ* is extended to *S* where $PQ = QS$. The point *U* divides the side *PR* in the ratio $3 : 2$. The point *T* is where the lines *QR* and *SU* cross.

Find the ratio $\dfrac{QT}{QR}$.



Solution

Construct *SV* parallel to *QR* to meet *PR* produced at *V*. (This is the tricky bit!)

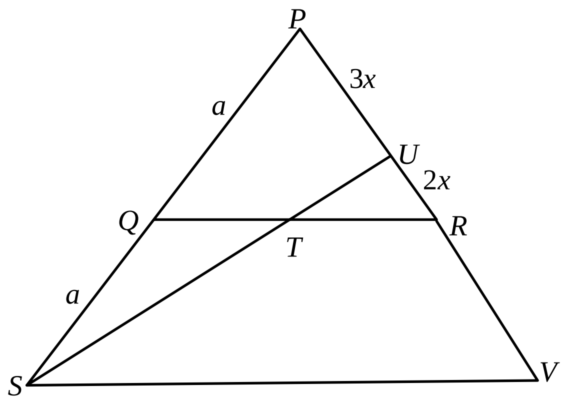
From similar triangles *PSV* and *PQR*, because $PS = 2PQ$, $PV = 2PR$. So $RV = 5x$.

Also $SV = 2QR$.

From similar triangles *UTR* and *USV*, $TR = \frac{2}{7}SV = \frac{4}{7}QR$.

Therefore $QT = QR - TR = \frac{3}{7}QR$.

i.e. $\dfrac{QT}{QR} = \dfrac{3}{7}$.



Alternative:

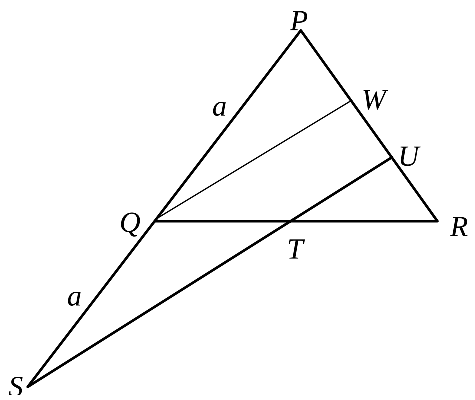
Draw *QW* parallel to *SU* to meet *PR* at *W*.

Then $\dfrac{QT}{TR} = \dfrac{WU}{UR}$ from triangles *QRW*, *TRU*.

Also $\dfrac{WU}{PU} = \dfrac{QS}{PS} = \dfrac{1}{2}$ from triangles *QWP*, *SUP*.

Hence $WU = \frac{1}{2}PU$.

So $\dfrac{QT}{TR} = \dfrac{PU}{2\,UR} = \dfrac{3}{4}$ and therefore $\dfrac{QT}{QR} = \dfrac{3}{7}$.



S2.

Dots are arranged in a rectangular grid with 4 rows and n columns. Consider different ways of colouring the dots, in which each dot either red or blue. A colouring is ‘good’ if no four dots of the same colour form a rectangle with horizontal and vertical sides.

Find the maximum value of n for which there is a good colouring.

Solution

First note that a grid with a repeated column has four dots of the same colour at the corners of a rectangle.

Consider a grid with only 3 rows. Then the only possible columns are

r r b b b r r b
r b r b r b r b
r b b r r r b b

The middle 6 columns form the largest good colouring.

Adding an extra row can never turn a bad colouring good, and so a good grid with 4 rows can have at most 6 columns.

r b b b r r
b r b r b r
b b r r r b
r r r b b b

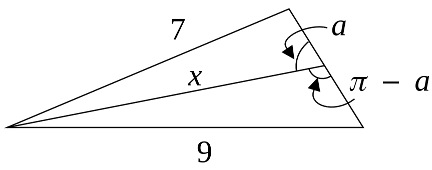
This example with 4 rows shows that a good grid with 6 columns is possible.

Hence 6 is the maximum value of n for which there is a good colouring.

S3.

A triangle has sides of length 4, 7 and 9 units. Find the length of the longest median. **Show your reasoning.**

Solution



The longest median bisects the shortest side. Let it have length x and make angle a (and $\pi - a$) with the shortest side.

Then using the cosine rule on the two small triangles

$$7^2 = x^2 + 2^2 - 2 \times 2x \cos a$$

$$9^2 = x^2 + 2^2 - 2 \times 2x \cos(\pi - a)$$

$$= x^2 + 2^2 + 2 \times 2x \cos a$$

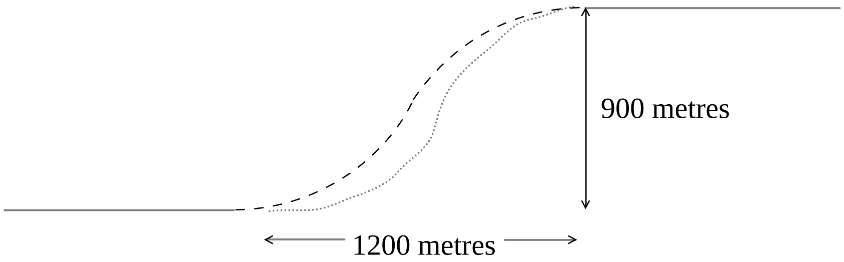
Adding

$$49 + 81 = 130 = 2x^2 + 2 \times 4$$

$$x^2 = 65 - 4 = 61$$

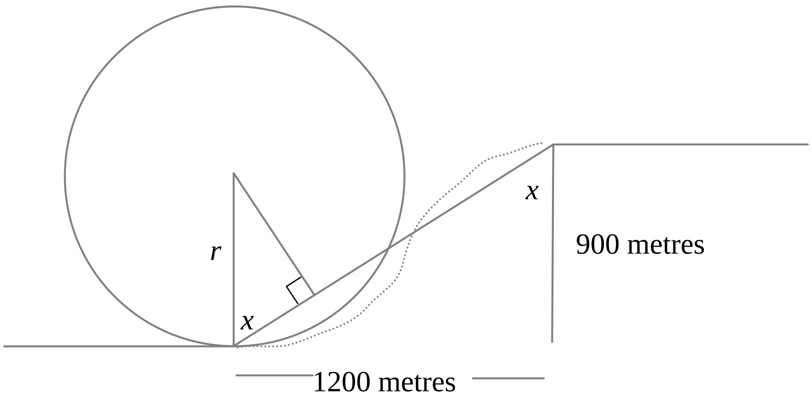
So the length of the longest median is $\sqrt{61}$.

S4.



Two straight sections of a road, each running from east to west, and located as shown, are to be joined smoothly by a new roadway consisting of arcs of two circles of equal radius. The existing roads are to be tangents at the joins and the arcs themselves are to have a common tangent where they meet. ′Find the length of the radius of these arcs.

Solution



$$\cos x = \frac{900}{1500} = \frac{3}{5}$$

$$\cos x = \frac{1500}{4}\,\frac{1}{r} = \frac{375}{r}$$

$$r = 375 \times \frac{5}{3} = 125 \times 5 = 625$$

i.e. the radius of the circles is 625 metres

S5.

Let

$$f(x) = x^n + a_1 x^{n-1} + \dots + a_n,$$

where $a_1, a_2, \dots, a_n$ are given numbers. It is given that $f(x)$ can be written in the form

$$f(x) = (x + k_1)(x + k_2)\dots(x + k_n).$$

By considering $f(0)$, or otherwise, show that $k_1 k_2 \dots k_n = a_n$.

Show also that

$$(k_1 + 1)(k_2 + 1)\dots(k_n + 1) = 1 + a_1 + a_2 + \dots + a_n$$

and give the corresponding result for $(k_1 - 1)(k_2 - 1)\dots(k_n - 1)$.

Hence, find the roots of the equation

$$x^4 + 22x^3 + 172x^2 + 552x + 576 = 0,$$

given that they are all integers.

Solution

$$f(0) = k_1 k_2 \dots k_n = a_n$$

$$f(1) = (1 + k_1)(1 + k_2)\dots(1 + k_n) = 1 + a_1 + a_2 + \dots + a_n$$

Hence

$$(k_1 + 1)(k_2 + 1)\dots(k_n + 1) = 1 + a_1 + a_2 + \dots + a_n$$

$$\begin{aligned} f(-1) &= (-1 + k_1)(-1 + k_2)\dots(-1 + k_n) \\ &= (-1)^n + (-1)^{n-1} a_1 + (-1)^{n-2} a_2 + \dots + a_n \end{aligned}$$

Hence

$$(k_1 - 1)(k_2 - 1)\dots(k_n - 1) = (-1)^n \left[1 - a_1 + a_2 - \dots + (-1)^n a_n\right]$$

The roots of the equation $f(x) = 0$ are $x = -k_1, -k_2, \dots, -k_n$

The roots of the given equation are all negative, because when $x = 0$ the LHS is 576, and it increases as x increases.

The product of the four k_i values for the LHS of this equation is $576 = 2^6 \times 3^2$

The product $(k_1 - 1)(k_2 - 1)\dots(k_n - 1) = (-1)^4\left[1 - 22 + 172 - 552 + 576\right] = 175 = 5^2 \times 7$ making it possible that the four values of $k_i - 1$ are 1, 5 , 5, 7.

This would mean that the 4 values of k_i were 2, 6, 6, 8, which fits with product of 576 found earlier.

Since $f(1) = (1 + 2)(1 + 6)(1 + 6)(1 + 8) = 1323$ and $f(-2) = (2 - 2) \times \dots = 0$ the function has the required values at four different values of x and hence will have the same values everywhere, in particular the same zeroes.

Thus we have the correct values for the k_i.

The roots of the given equation are therefore $-2, -6, -6, -8$.