2013-2014 Senior Solutions Round 1

S1

On a tiny remote island where the death sentence still exists a man can be granted mercy after receiving the death sentence in the following way:

- he is given 18 white balls and 6 black balls.
- he must divide them between three boxes with at least one ball in each box.
- he is then blindfolded and must choose a box at random and then a single ball from within this box.

He receives mercy only if the chosen ball is white.

Find the probability that he receives mercy when he distributes the balls in the most favourable manner.

Solution

If he puts all the black balls in one box and the whites in the other two the probability of reprieve is $\frac{2}{3}$.

If he distributes the balls evenly (6w and 2b in each box) the probability of reprieve is $\frac{3}{4}$ which is better.

For a better chance, he should make sure that two of the boxes contain only white balls, one in each.

So he should put all the black balls in the third box and then he should put all but two white balls in the third box as well to improve his chances there.

So he should be certain of reprieve with 2 boxes (1w in each) and put the remaining balls in the third (16w and 6b). Then the probability is

$$\frac{1}{3} + \frac{1}{3} + \frac{1}{3} \times \frac{16}{22} = \frac{10}{11}.$$

S2

Three types of item, A, B and C, are for sale. Items of type A sell at 8 for £1. Items of type B sell for £1 each. Items of type C sell for £10 each. A selection of 100 items which includes at least one of each type costs £100. How many items of type B are there in the selection?

Solution

Let a be the number of items of type A in the selection.

Let b be the number of items of type B in the selection.

Let c be the number of items of type C in the selection.

$$\frac{a}{8} + b + 10c = 100$$

$$a + b + c = 100$$

So

$$b = 100 - a - c$$

$$\frac{a}{8} + 100 - a - c + 10c = 100$$

$$-\frac{7}{8}a + 9c = 0$$

$$7a = 72c$$

Either $a = c = 0$, which would mean that there were item types missing from the selection, and hence is not possible.

Or $a = 72$ and $c = 7$ (larger multiples of these are not possible as there are only 100 items in all.)

Hence $b = 100 - 72 - 7 = 21$

i.e. there are 21 items of type B.

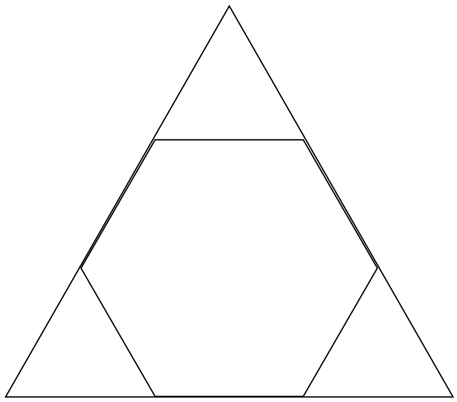
S3

Prove that the area of a regular hexagon with side of length a is $\frac{3\sqrt{3}}{2}a^2$.

Not all equiangular hexagons are regular: find a formula for the area of an equiangular hexagon with sides of length a, b, a, b, a, b in that order.

Solution

Consider the regular hexagon inscribed within an equilateral triangle as shown:

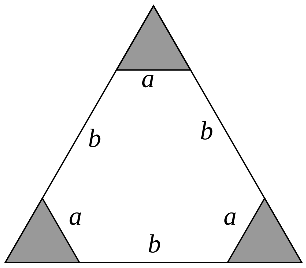


Area of hexagon = area of large triangle − 3 × area of small triangle

$$= \tfrac{1}{2} \times 3a \times 3a \times \sin 60° - 3 \times \tfrac{1}{2} \times a \times a \times \sin 60°$$

$$= \left(\frac{9}{2} - \frac{3}{2}\right) a^2 \times \frac{\sqrt{3}}{2}$$

$$= \frac{3\sqrt{3}}{2} a^2$$



Starting with the hexagon, add equilateral triangles of side-length a to each side of the hexagon of length a. This creates an equilateral triangle of side-length $(2a + b)$.

Area of (a,b) hexagon = area of large triangle − 3 × area of small triangle

$$= \frac{1}{2} \times (2a + b) \times (2a + b) \times \sin 60° - 3 \times \frac{1}{2} \times a \times a \times \sin 60°$$

$$= \left(\frac{(2a + b)^2}{2} - \frac{3a^2}{2}\right) \times \frac{\sqrt{3}}{2} = \frac{\sqrt{3}}{4}\left(a^2 + 4ab + b^2\right).$$

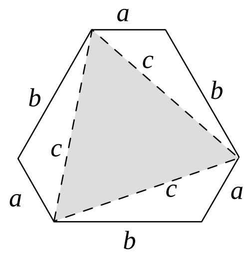
Alternative solution

A marker noticed the following solution and it is definitely worthy of wider circulation. It has been slightly polished.

We join alternate vertices of the hexagon to create the dissection shown below.

We are given that the hexagon is equiangular so the three peripheral triangles are congruent (two sides and the included angle). We denote the third side of each by c.

The shaded triangle is therefore equilateral.



Consider one of the peripheral triangles. We know the obtuse angle is 120° so we can use the trigonometric formula to state:

$$\text{Area} = \frac{1}{2}ab\sin 120^\circ = \frac{1}{2}ab \times \frac{\sqrt{3}}{2} = \frac{\sqrt{3}}{4}ab.$$

Also, making use of the Cosine Rule,

$$c^2 = a^2 + b^2 - 2ab\cos 120^\circ = a^2 + b^2 - 2ab \times \left(-\frac{1}{2}\right) = a^2 + b^2 + ab.$$

Now consider the shaded triangle. We once again use the trigonometric formula to state:

$$\text{Area} = \frac{1}{2}c^2\sin 60^\circ = \frac{1}{2}c^2 \times \frac{\sqrt{3}}{2} = \frac{\sqrt{3}}{4}c^2 = \frac{\sqrt{3}}{4}\left(a^2 + b^2 + ab\right).$$

Thus, the area of the hexagon is given by:

$$\begin{aligned} 3 \times \left(\frac{\sqrt{3}}{4}ab\right) + \frac{\sqrt{3}}{4}\left(a^2 + b^2 - ab\right) &= \frac{\sqrt{3}}{4}\left(3ab + a^2 + b^2 + ab\right) \\ &= \frac{\sqrt{3}}{4}\left(a^2 + b^2 + 4ab\right). \end{aligned}$$

S4

Naismith's rule enables hillwalkers to estimate route times according to the following rule (updated from its original formulation in 1892): allow one hour for every 5 km measured on the map, plus one hour for every 600 m of ascent.

A group of hillwalkers was confronted by an exactly conical hill with diameter 4 km rising from a plain. Some of the group chose to climb to the summit of the hill, whereas the rest of the group chose to walk on the flat round its circular base. Neither group stopped en route, and the summit group arrived at the meeting point on the opposite side of the hill 20 minutes after the walk round group.

Assuming Naismith's rule is correct, determine the height of the hill above the plain. Give your answer to the nearest metre.

Solution

Let the height be h km. So the time taken by the group who go via the summit is

$$\frac{4}{5} + \frac{h}{0{\cdot}6} = \frac{4}{5} + \frac{5h}{3}.$$

The distance round is half the circumference, i.e. $\frac{1}{2}4\pi = 2\pi$, so the time taken by the group which goes around the base is

$$\frac{2\pi}{5}.$$

Hence

$$\frac{4}{5} + \frac{5h}{3} = \frac{2\pi}{5} + \frac{1}{3}$$

$$12 + 25h = 6\pi + 5$$

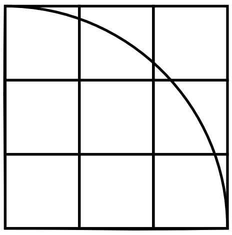
$$h = \frac{6\pi - 7}{25} \approx 0{\cdot}474.$$

The height of the hill is about 474 metres.

S5
A cube of edge 6 cm is divided into 216 unit cubes by planes parallel to the faces of the cube. A sphere of diameter 6 cm sits inside the cube so that the faces of the cube are tangents to the sphere. How many complete unit cubes are contained within this sphere?

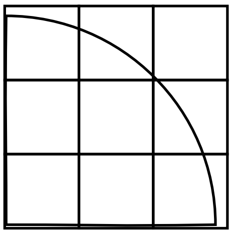
Solution
Assume the centres of the cube and sphere are at the origin, and consider the positive octant.

For the small cubes lying on $z = 0$.



At (2,2,1), $x^2 + y^2 + z^2 = 4 + 4 + 1 = 9$ so this point is on/within the sphere. Altogether 4 cubes in this layer lie within the sphere.

For the small cubes lying on z = 1.



At (2,1,2), $x^2 + y^2 + z^2 = 4 + 4 + 1 = 9$ so this point is on/within the sphere. Altogether 3 cubes in this layer lie within the sphere.

No small cubes lying on $z = 2$ are completely within the sphere.

Hence there are 7 small cubes within the sphere in this octant
i.e. $7 \times 8 = 56$ in all.