2012-2013 Senior Set 2 solutions

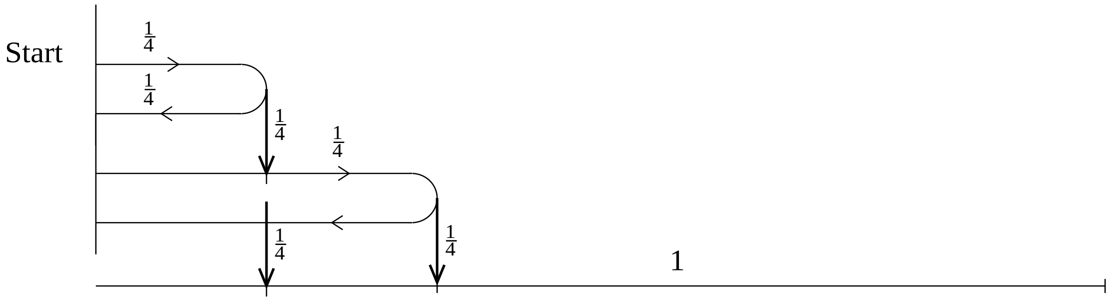
S1. Show that the maximum range of an aeroplane is extended by a factor of $\frac{1}{3}$ when there is a second identical support plane which sets off at the same time to provide in-air refuelling. The support plane must return safely to the starting point.

Now consider the situation where there are two identical support planes which can (instantaneously) refuel each other or the original plane as required. The support planes set off at the same time as the original plane and both must return safely to the starting point. By how much can the maximum range of the plane be extended?

Solution 1

Both planes fly $\frac{1}{3}$ of their maximum range, using $\frac{1}{3}$ of their fuel capacity, and then the support plane transfers $\frac{1}{3}$ of its fuel capacity to the first plane. The support plane uses the remaining $\frac{1}{3}$ of its fuel capacity to return safely to base and the first plane is full of fuel. Thus the range of the first plane has been extended by $\frac{1}{3}$.

All three planes fly $\frac{1}{4}$ of their maximum range, and then one support plane refuels each of the others with $\frac{1}{4}$ of its fuel capacity, leaving itself with $\frac{1}{4}$ to fly home and each of the other planes full to capacity. The two remaining planes fly another $\frac{1}{4}$ of their maximum range before the second support plane transfers $\frac{1}{4}$ of its fuel capacity to the long distance plane, leaving itself with $\frac{1}{2}$ of its fuel capacity with which to return safely and the long distance plane full of fuel. Thus the range of the first plane has been extended by $\frac{1}{2}$.



Solution 2

Let the original range of the plane be r.

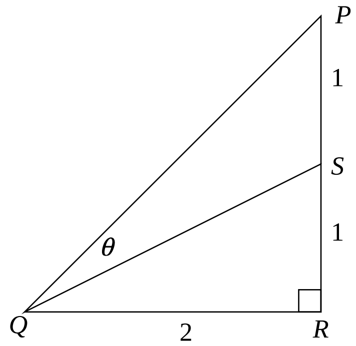
With two planes, let the support plane fly x out and x back, and let the main plane fly x out and then a further y. The total distances flown by the two planes must be $2r$, so

$$3x + y = 2r.$$

To maximise $x + y$ subject to this constraint we must make y as large as possible. To do this, let the second plane be full after refuelling, so that $y = r$. This makes $x = \frac{1}{3}r$, so the extension is $x + y - r = \frac{1}{3}r$, or $\frac{1}{3}$ of the original range.

With three planes, let the first support plane fly x out, then y out, then $y + x$ back, and let the main plane fly x out, then y out, then z out. As before $5x + 3y + z = 3r$. To maximise $x + y + z$ subject to this contraint we must maximise z, so we make $z = r$, and we must then maximise y, so $x + 3y + z = 2r$ (that is, the second support plane and the main plane are both full after the first refuelling). This makes $x = y = \frac{1}{4}r$, so the extension is $x + y + z - r = \frac{1}{2}r$, or $\frac{1}{2}$ of the original range.

S2. In the diagram angle PRQ is a right angle and PS and SR are both length 1 cm. QR is length 2cm. Find the exact value of $\tan\theta$.



Solution

Let us denote $\angle PQR$ by a and $\angle SQR$ by b.
Triangle PQR is isosceles, and so $\angle PQR = 45°$.

$$\tan a = 1$$

From the diagram

$$\tan b = \frac{1}{2}$$

$$\begin{aligned}\tan\theta &= \tan(a - b) \\ &= \frac{\tan a - \tan b}{1 + \tan a \times \tan b} \\ &= \frac{1 - \frac{1}{2}}{1 + 1 \times \frac{1}{2}} \\ &= \frac{1}{3}.\end{aligned}$$

S3. Four apparently identical small objects have weights a, b, c and d such that

$$a < b < c < d.$$

It is also known that

$$a > \frac{2}{3}b$$

and

$$c < \frac{3}{4}d.$$

Using only a balance, show how it is possible to pick out the heaviest object in just 2 weighings.

Solution

The first weighing must involve 2 objects in each pan. Clearly

$$a + b < c + d$$

and

$$a + c < b + d$$

How does $a + d$ relate to $b + c$? Try some values that nearly satisfy the conditions:

$$a = 2,\ b = 3,\ c = 3,\ d = 4$$

a and d must both weigh a little more than the values here, making

$$a + d > b + c$$

Can we prove this in general?

$$a + d > \frac{2}{3}b + \frac{4}{3}c = \frac{2}{3}b + \left(\frac{1}{3}c + c\right) > \frac{2}{3}b + \frac{1}{3}b + c = b + c$$

Thus the pan containing object d will always be the heavier.

The second weighing puts these two objects in separate pans, and the heavier will be weight d.

S4. I was reading a novel aimed at children and noticed that exactly half of the page numbers began with ‘1’. Work out the number of pages in the novel.

Solution

We would expect a children’s novel to have between 100 and 1000 pages (even Harry Potter does not exceed 1000).

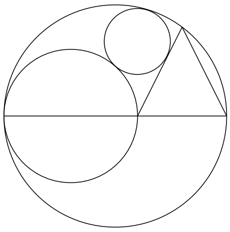
99 pages would include 11 with initial 1 (1 itself and then10-19).

100 pages would include 12 and so on up to 199.

The number of initial 1s is 88 less than the number of pages, so the first possible answer is $2 \times 8 = 176$ pages

After 199, the number of initial 1s stays constant 111 (11 plus 100−199) so the second possible answer is $2 \times 111 = 222$ pages

S5. Instructions for drawing this diagram are as follows:

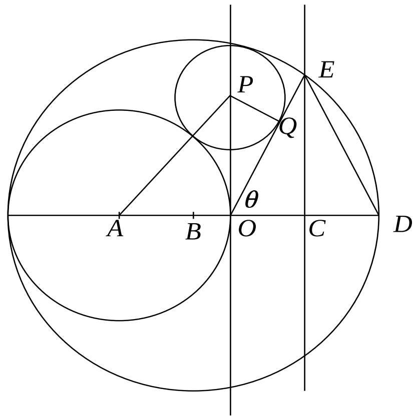


Split the diameter of the large circle into two parts. On one part, draw a second circle with that part as diameter. On the other part, draw an isosceles triangle with that part as base and the other vertex on the circumference of the large circle. Now draw a third circle so that it touches the other two circles and the triangle, as shown in the diagram.

Show that the centre of the third circle must lie on the line perpendicular to the diameter of the large circle and meeting this diameter at the point where the second circle and triangle touch.

Solution

[The diagram shows the case where the diameter of the smaller circle is greater than the radius of the large circle. In the other case a similar argument to that given below applies with an altered diagram.]



In the figure with additional letters as show (B is the centre of the largest circle and A the centre of the middle-sized circle) let $|BD| = R$ and $|AO| = r$. We must show that there is a point P on the line through O perpendicular to AD satisfying the equations

$$|PA| - r = R - |PB| = |PQ|, \tag{1}$$

where Q is the foot of the perpendicular from P to OE. To do this, let $|OP| = z$; we can then work out the three quantities $|PA| - r$, $R - |PB|$ and $|PQ|$ as follows.

Since $|AO| = r$, applying Pythagoras' theorem to the triangle AOP gives

$$|PA| - r = \sqrt{|AO|^2 + |OP|^2} - r = \sqrt{r^2 + z^2} - r. \tag{2}$$

Since $|BD| = R$ and $|OD| = 2R - 2r$, it follows that $|BO| = |R - 2r|$. Applying Pythagoras' theorem to the triangle BOP gives

$$R - |PB| = R - \sqrt{|BO|^2 + |OP|^2} = R - \sqrt{(R - 2r)^2 + z^2}. \tag{3}$$

From the similar triangles POQ and OEC one sees that

$$|PQ| = \frac{|OC|\,|OP|}{|EO|} = \frac{(R - r)z}{|EO|}.$$

Since $|BC| = |BD| - |CD| = R - (R - r) = r$, applying Pythagoras' theorem to OCE and BCE shows that

$$|EO| = \sqrt{|OC|^2 + |CE|^2} = \sqrt{|OC|^2 + |BE|^2 - |BC|^2} = \sqrt{(R - r)^2 + R^2 - r^2},$$

hence

$$|PQ| = \frac{(R - r)z}{\sqrt{(R - r)^2 + R^2 - r^2}} = \frac{(R - r)z}{\sqrt{2R(R - r)}} = \sqrt{\frac{R - r}{2R}}\,z. \tag{4}$$

Because of (2)-(4), to find a point P satisfying (1) it suffices to fine a number z such that

$$\sqrt{r^2 + z^2} - r = R - \sqrt{(R - 2r)^2 + z^2} = \sqrt{\frac{R - r}{2R}}\,z.$$

Solve these equations to show that there is a solution given by

$$z = \frac{2r\sqrt{2R(R - r)}}{R + r}.$$

Hence the centre P of the smallest circle does lie on the lie PO which is perpendicular to the diameter of the largest circle, as required.