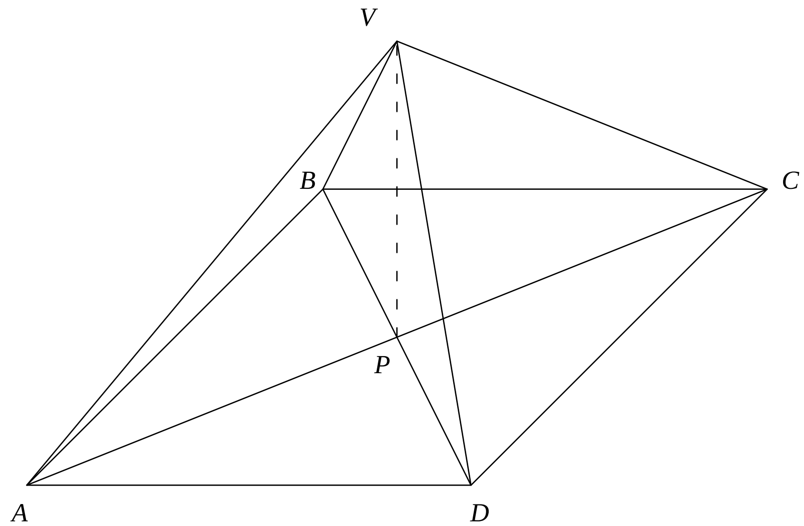
2012-2013 Senior Solutions Round 1

S1

Four spheres each of radius 10 cm lie on a horizontal table so that the centres of the spheres form a square of side 20 cm. A fifth sphere of radius 10 cm is placed on them so that it touches each of the spheres without disturbing them. How far above the table is the centre of the fifth sphere?

Solution



In the diagram, *A*, *B*, *C*, *D* and *V* are the centres of the 5 spheres. The shape *ABCDV* is a square-based pyramid, with all 8 of its edges equal, in this case each edge is 20 cm.

The base is a square so $\triangle ABC$ is a right-angled triangle and we can use Pythagoras' Theorem to get

$$AC^2 = 20^2 + 20^2 = 2 \times 400$$

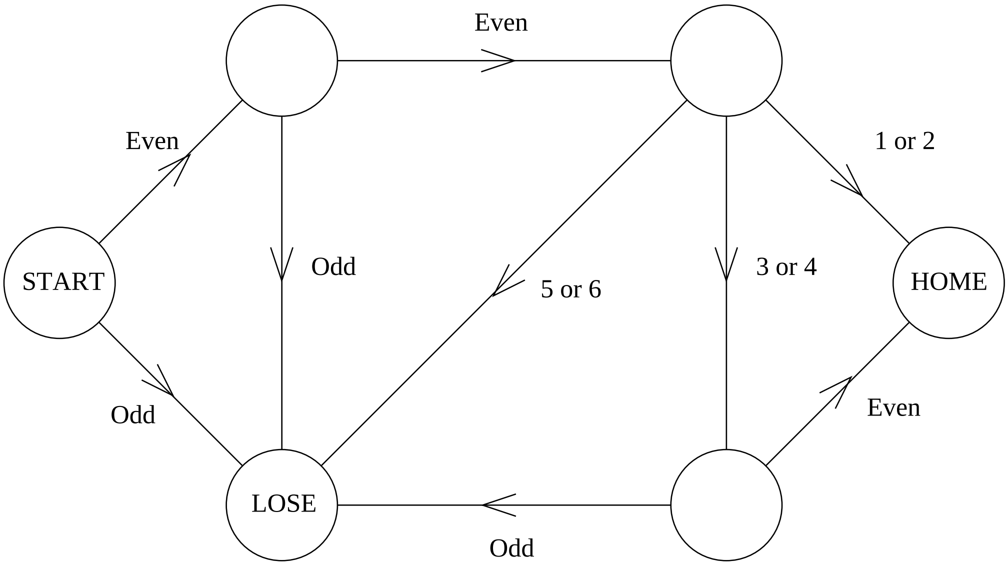
$$\Rightarrow AC = 20\sqrt{2}.$$

But, since the sides of $\triangle VAC$ are 20, 20, $20\sqrt{2}$, we can apply the converse of Pythagoras' Theorem to prove that $\triangle VAC$ is a right-angled triangle. As this triangle is congruent with, for example, $\triangle ABC$, we can say that the height VP is equal to half of a diagonal of the base. Thus we get

$$VP = 10\sqrt{2}.$$

But, the plane *ABCD* is 10 cm above the table so the height of the centre of the fifth sphere above the table is $10\left(1 + \sqrt{2}\right)$ cm.

S2.



A counter placed on the start circle moves in the direction determined by the throw of a normal six-sided die. What is the probability of reaching the HOME circle?

Solution

There are two routes from START to HOME.

For the fast track, the probability is $\frac{1}{2} \times \frac{1}{2} \times \frac{1}{3} = \frac{1}{12}$.

For the other track, the probability is $\frac{1}{2} \times \frac{1}{2} \times \frac{1}{3} \times \frac{1}{2} = \frac{1}{24}$.

The total probability is therefore $\frac{1}{12} + \frac{1}{24} = \frac{1}{8}$.

S3.

Crispin had been learning about perfect squares. He had then taken some of the number blocks borrowed from his little sister Suzy and arranged them to form five numbers, exactly one of which was a perfect square. He then went off to fetch his father to see if he could work out which of the numbers he had formed was the perfect square.

When they returned, however, they discovered that Suzy had been playing with the number blocks again. She had rearranged the blocks in the first number and also in the second number. In addition, she had taken away all but the last two of the blocks in the third number, all but the last block in the fourth number and all the blocks in the fifth number. This is what the remaining blocks looked like:

(1)	3 5 5 8 7 3 2
(2)	7 3 7 4 3 7 3
(3)	7 5
(4)	8
(5)	

Crispin's father looked at the blocks and said, "Never mind, I can still tell you which number was the perfect square". Can you?

Solution

The square of an integer must end in 0, 1, 4, 5, 6 or 9.

If it ends in 5 it has the form $(10a + 5)^2 = 100\left(a^2 + a\right) + 25$ and so must end in 25.

Thus numbers (3) and (4) cannot have been squares.

If (1) was a square it could only end in 25.

Suppose it was of the form $(100a + 10b + 5)^2 = 1000\left(10a^2 + 2ab + a\right) + 100\left(b^2 + b\right) + 25$ so that the hundreds digit must be the units digit from $b^2 + b = b(b + 1)$.

But checking all the numbers for b from 0 to 9 shows that the hundreds digit must be 0, 2 or 6. Since we have already used the "2" for the 25, the number (1) cannot be a square.

If (2) was a square, then it must end in 4 and so the number being squared must have the form $(10a + 2)$ or $(10a + 8)$.

Squaring these gives $100a^2 + 10 \times 4a + 4$ or $100a^2 + 10(16a + 6) + 4$.

Thus the tens digit must be even and as there are no even digits apart from 4 in (2) it cannot be a square.

Thus it must have been (5) that was the square.

S4.

Consider the sequence of all positive integers for which the sum of the digits is divisible by 7, arranged in order of increasing magnitude.

(a) Write down the first members of this sequence less than 100.

(b) What is the maximum difference between consecutive members of the whole sequence? Justify your answer.

Solution

(a) The numbers less than 100 which satisfy the stated property are

7, 16, 25, 34, 43, 52, 59, 61, 68, 70, 77, 86, 95

(b) For each number of hundreds, there are two or more overlapping sequences, with sum of digits 7, sum of digits 14, sum of digits 21, …

The difference between successive members is 9, or less where two sequences overlap.

Any gaps larger than 9 will occur when a whole number of hundreds is passed. These points are

…, 95, 106, …
…, 194, 205, ...
…, 293, 304, …
…, 392, 399, 403, ...
…, 498, 502, ...
…, 597, 601, ...
…, 696, 700, ...
…, 795, 806, ...
…, 894, 905, ...
…, 993, 1006, …

This last gap is 13, the largest so far.

However if the gap were 14 it would be possible to either add 7 to the smaller number and stay within the sequence, or subtract 7 from the larger number and also stay within the sequence. Thus the largest possible gap is 13.

Alternative for (b)

Let a and b be consecutive members of the sequence.

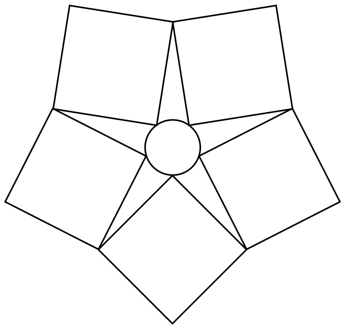
Let d be the last digit of a and let $a = 10k + d$.

If $d \leqslant 2$ then $b = a + 7$.

If $d \geqslant 3$ then $b = 10(k + 1) + d'$ with $0 \leqslant d' \leqslant 6$. The largest conceivable difference between a and b is therefore 13, which will happen if $d = 3$ and $d' = 6$.

The difference will actually be 13 if k and $k + 1$ have a total of digits with remainders 4 and 1 when divided by 7, which can be achieved by $k = 9\ldots 9$, $k + 1 = 10\ldots 0$ for a suitable number of 9s and 0s. One suitable number of 9s and 0s is two, so that $a = 993$, $b = 1006$.

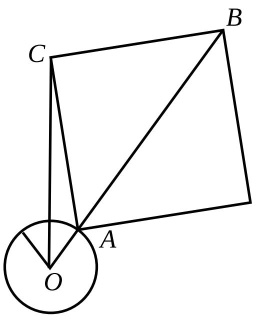
S5.



In the diagram, five identical squares are arranged symmetrically round a circle so that their vertices touch each other and one vertex of each square lies on the circle.

Starting with a circle with radius equal to the length of a side of the squares, how many squares would it take to be similarily arranged?

Solution 1



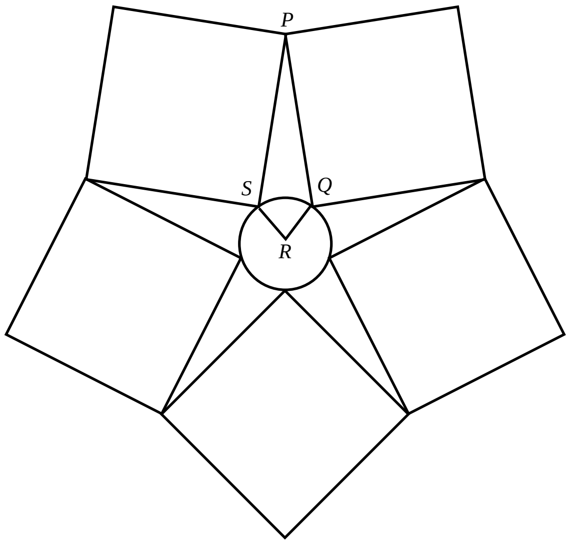
The radius through the point of contact of the square and the circle extends to be the diagonal of the square.

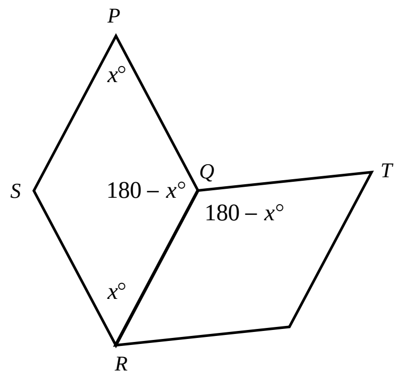
If there are n squares round the circle, then this radius and the radius through the point of contact of two squares make an angle of $180° \div n$.

Thus in the diagram $\angle AOC = 180°/n$. Also $\angle BAC = 45°$.

If $OA = AC$ then $\angle AOC = \angle OCA$. Now $\angle BAC = \angle AOC + \angle OCA$. So $\angle AOC = \frac{1}{2} \times 45° = \frac{1}{8} \times 180°$. So there are 8 squares round the circle.

Solution 2





Consider the quadrilateral *PQRS*. When the radius is equal to the length of the side of a square, *PQRS* will be a rhombus. Let $\angle QRS = x°$. It follows that $\angle PQR = (180 - x)°$. In addition, $\angle PQT = 90°$ as it is the corner of a square. So, at the point *Q*,

$$2\,(180 - x)° + 90° = 360°$$

$$180 - x + 45 = 180$$

$$\text{So } x = 45.$$

Thus, since $360 \div 45 = 8$, we have shown there will be eight squares round the circle.