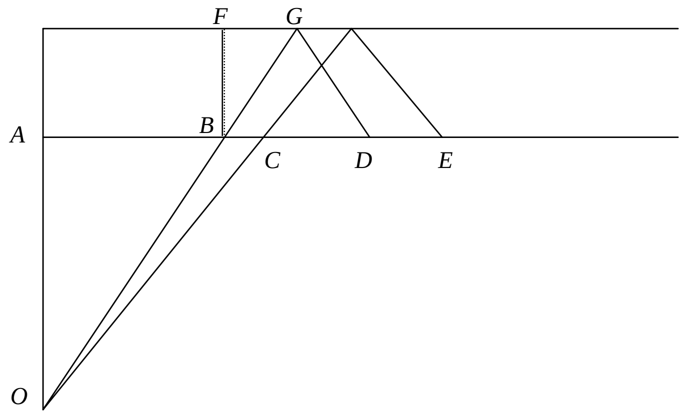
2011-12 Senior Set 2 solutions

S1. Andy is standing at a bus stop near his house. Through a small window, he can see the reflection of a television in a large mirror. The television set is mounted on the same wall of the house as the window and the mirror is on the opposite wall. He also notices that the reflection he sees through the small window is the full width of the TV but no more.

He wonders how wide his neighbour's TV is. But as the house is exactly like his own he can work it out. The small window is 50 cm wide and the room is 4 m deep. Furthermore he is exactly 10m from the nearest point on the front wall of the house on which the window and the TV are. How does he calulate this and what is the width of the TV?

Solution

We draw a diagram to represent the situation and solve the problem.



In this diagram, Andy is at O, the window is BC and the TV is DE. Note that $OA = 10$ m and $BF = 4$ m.

Let the length of AB be x m.

By similar triangles, $\frac{FG}{FB} = \frac{AB}{OA}$ so $FG = \frac{4}{10}x$.

So $BD = 2 \times FG = \frac{8}{10}x$ and $AD = x + \frac{8}{10}x = \frac{18}{10}x$.

Note that $AC = x + \frac{1}{2}$ and exactly the same method gives that $AE = \frac{18}{10}\left(x + \frac{1}{2}\right)$.

Thus the width of the TV is $DE = AE - AD = \frac{9}{10}$, i.e. the TV is 0.9 m wide.

S2. An employee submitted a claim for travelling expenses. The accountant dealing with the claim made a mistake in which he interchanged the numbers of pounds and pence. This resulted in a greater payment being made than was due. Not noticing the error, the employee spent £3.19 on a magazine from the payment on his way home. When he checked his cash, he found that the sum he now had was exactly 7/5 of his original claim. How much did he claim for travelling expenses?

Solution

Suppose the original claim was £x and y pence i.e. $(100x + y)$ pence.

He received £y and x pence i.e. $(100y + x)$pence.

Let all calculations now be in pence. So he was overpaid $99(y - x)$.

Thus $99(y - x) - 319 = \frac{2}{5}(100x + y)$, i.e.$493y - 695x = 5 \times 319$.

Clearly y is a multiple of 5.

Note also that $493 = 17 \times 29$ and $319 = 11 \times 29$.

So x is a multiple of 29.

Let $y = 5Y$ and $x = 29X$ to obtain $17Y - 139X = 11$.

Note that $x < 100$ so X can only take the values 1, 2 or 3.

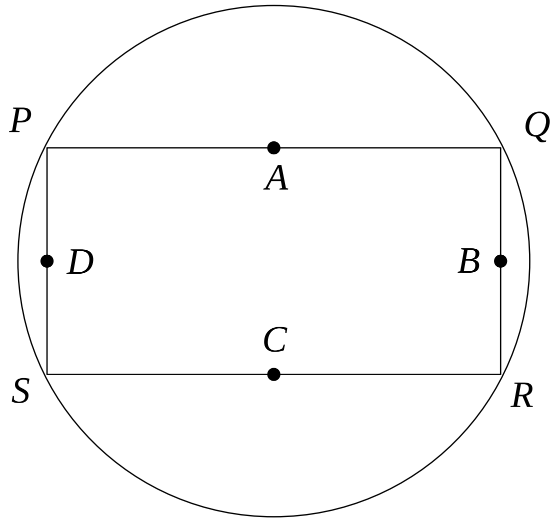
If $X = 1$ then 139 + 11 = 150 is not a multiple of 17.

If $X = 2$ then 278 + 11 = 289 = 17 × 17 so $Y = 17$.

If $X = 3$ then 417 + 11 = 428 is not a multiple of 17.

So $x = 58$ and $y = 85$ and the original claim was for £58.85.

S3. The vertices of a rectangle $PQRS$ lie on a circle of radius r. The points A, B, C and D are the midpoints of PQ, QR, RS and SP respectively. Determine, with proof, the greatest possible area of the quadrilateral $ABCD$.



Solution

Let the centre of the circle be O, and let $\angle QSR = x$. Then

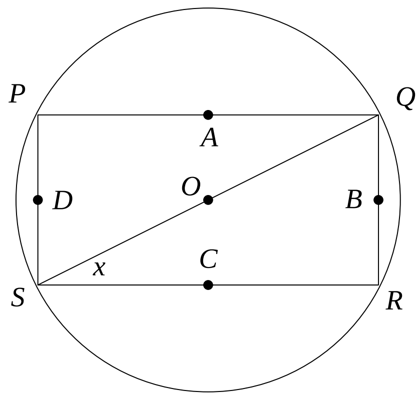
$$\text{area}\, PQRS = (2r\cos x).(2r\sin x) = 2r^2 \sin 2x.$$

Triangles AQB and AOB are congruent and so have the same area, and similarly at each corner of rectangle $PQRS$. Thus the area of $ABCD$ is half the area of rectangle $PQRS$, so

$$\text{area}\, ABCD = r^2 \sin 2x.$$

Since the maximum value of $\sin 2x$ is 1, the maximum area of $ABCD$ is equal to r^2.

(In this case $2x = 90°$ so $x = 45°$. This means that $PQRS$ and $ABCD$ are squares.)

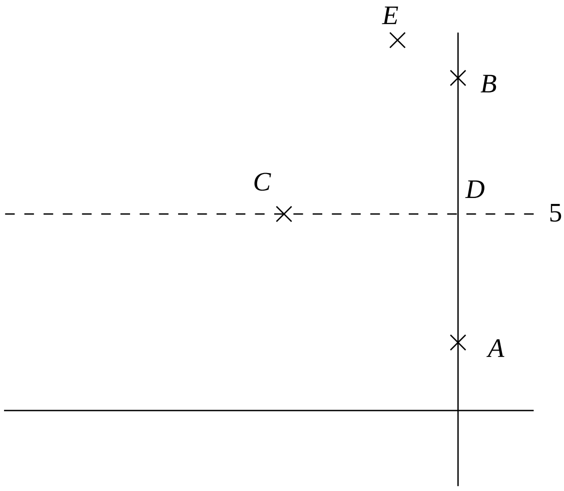


S4. Determine the equation of the circle which satisfies these conditions:

- the graph is tangential to the x-axis;
- it passes through $(-3, 10)$ and $(0, 1)$;
- the two points where the graph intersects the positive y-axis are 8 units apart.

Solution 1

Let A be $(0, 1)$. Then since it also passes through $E = (-3, 10)$, using the last statement means that the circle passes through $B(0, 9)$. Bisecting this chord AB gives the y coordinate of the centre, C, as 5.



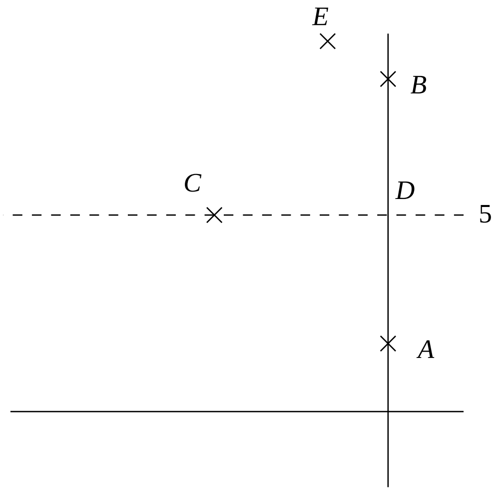
Since the circle is tangential to the x-axis the radius is 5 units.

Using Pythagoras' theorem in triangle ACD gives CD as 3 units meaning the centre of the circle is $(-3, 5)$ producing the equation

$$(x + 3)^2 + (y - 5)^2 = 25$$

Solution 2

The circle meets the positive y-axis at two points which are 8 units apart. One of these points is $A(0, 1)$, so the other is $B(0, 9)$. The centre of the circle must lie on the perpendicular bisector of AB and this is the line $y = 5$, so the equation has the form



$$(x - a)^2 + (y - 5)^2 = r^2.$$

Since the circle passes through $(-3, 10)$ and $(0, 1)$ we get

$$(-3 - a)^2 + 25 = r^2$$

$$(-a)^2 + 16 = r^2$$

Subtracting these gives

$$9 + 6a + a^2 + 25 - \left(a^2 + 16\right) = 0 \Rightarrow 6a + 18 = 0 \Rightarrow a = -3 \Rightarrow r^2 = 25 \Rightarrow r = 5.$$

So the equation of the circle is $(x + 3)^2 + (y - 5)^2 = 25$.

S5. In how many ways can the number 1 000 000 be expressed as the product of three positive integers a, b, c where $a \leqslant b \leqslant c$?

Solution

Version 1

$1\,000\,000 = 10^6 = 2^6 5^6 = 2^{p+q+r}5^{s+t+u} = (2^p5^s)(2^q5^t)(2^r5^u)$

where p, q, r and s, t, u can be

0, 0, 6   0, 1, 5   0, 2, 4   0, 3, 3   1, 1, 4   1, 2, 3   2, 2, 2

in any order.

Considering each of the possibilities for p, q, r we can determine how many solutions each set of s, t, u gives.

For example, $(2^1\ldots)(2^2\ldots)(2^3\ldots)$ gives 6 solutions if combined with $5^1, 5^2, 5^3$ in some order.

The number of solutions for different combinations of p, q, r and s, t, u is given in the table below.

	3 2 1	0 1 5	0 2 4	0 0 6	0 3 3	1 1 4	2 2 2	total
3 2 1	6	6	6	3	3	3	1	28
0 1 5	6	6	6	3	3	3	1	28
0 2 4	6	6	6	3	3	3	1	28
0 0 6	3	3	3	2	2	2	1	16
0 3 3	3	3	3	2	2	2	1	16
1 1 4	3	3	3	2	2	2	1	16
2 2 2	1	1	1	1	1	1	1	7
								139

Version 2

We count triples $(2^p5^s, 2^q5^t, 2^r5^u)$ such that p, q, r, s, t, u are non-negative integers with

$p + q + r = 6$ and $s + t + u = 6$,

$p \leqslant q \leqslant r$,

if $p = q$ then $s \leqslant t$,

if $q = r$ then $t \leqslant u$.

The possible values for $p\,q\,r$ are: 0 0 6; 0 1 5; 0 2 4; 0 3 3; 1 1 4; 1 2 3; 2 2 2.

If $p\,q\,r$ is 0 1 5 or 0 2 4 or 1 2 3 then there are 28 possible values for $s\,t\,u$, given by

006, 015, 024, 033, 042, 051, 060,
105, 114, 123, 132, 141, 150,
204, 213, 222, 231, 240,
303, 312, 321, 330,
402, 411, 420,
501, 510,
600.

If $p\ q\ r$ is 0 0 6 or 1 1 4 then there are 16 possible values for $s\ t\ u$, given by

006,	015,	024,	033,	042,	051,	060,
114,	123,	132,	141,	150,		
222,	231,	240,				
330.						

If $p\ q\ r$ is 0 3 3 then there are similarly 16 possible values for $s\ t\ u$.

If $p\ q\ r$ is 2 2 2 then there are 7 possible values for $s\ t\ u$, given by

006,	015,	024,	033,
114,	123,	222.	

This gives a total of $3 \times 28 + 3 \times 16 + 7 = 139$.

Version 3

The problem is equivalent to writing 10^6 as the product of 2^p5^s, 2^q5^t and 2^r5^u where p, q, r, s, t and u are integers between 0 and 6 inclusive and $p + q + r = 6$ and $s + t + u = 6$.

Each of the two sums corresponds to a partition of 6 into the sum of three non-negative integers. If order is taken into account there are 28 such partitions. A nice way to see this is to think of 8 slots into which we insert 6 1s and two dividing lines. The two dividers can be placed in ${}^8C_2 = 28$ ways. Each such placing gives rise to a partition of the required type and conversely each required partition corresponds to such a placing. For example, if we call the dividers X and Y then

1 1 X 1 1 1 Y 1 corresponds to 2 + 3 + 1
X 1 1 1 1 Y 1 1 corresponds to 0 + 4 + 2.

To count the number of possible factorisations, we must avoid repetitions. One way to do this is to use unordered partitions for p, q, r. We can list these, along with the number of possible permutations of each, with commas omitted for convenience.

Partition	0 0 6	0 1 5	0 2 4	0 3 3	1 1 4	1 2 3	2 2 2
Number of Perms	3	6	6	3	3	6	1

(This confirms that the total number of ordered permutations is 28 as before.)

Having used unordered permutations for p, q, r we can use ordered permutations for s, t, u.

There are three cases.

(i) Each of 0 1 5, 0 2 4 and 1 2 3 (for $p\ q\ r$) can be paired with any of the 28 possible ordered permutations for $s\ t\ u$. This gives $3 \times 28 = 84$.

(ii) Taking each of 0 0 6, 0 3 3 and 1, 1, 4 (for $p\ q\ r$) will produce repetitions.

* Consider 0 0 6.

If we look at permutations of 0 1 5 (for $s\ t\ u$) then each of the pairs

0 1 5 and 1 0 5; 0 5 1 and 5 0 1; 1 5 0 and 5 1 0

will produce the same factorisation. So we can only take one of each pair and therefore lose 3 choices.
Similarly we lose 3 choices when we take permutations of 0 2 4 or 1 2 3 for $s\ t\ u$.

Next look at permutations of 0 0 6 for $p\ q\ r$. Each of 0 6 0 and 6 0 0 will produce the same factorisation.
So we lose 1 choice.

The same applies with permutations of 0 3 3 and 1 1 4.

So in total we lose 3 + 3 + 3 + 1 + 1 + 1 = 12 leaving 28 − 12 = 16 choices.

* Similar considerations apply to 0 3 3 and 1 1 4 (for $p\ q\ r$).
So the number for factorisations arising from this case is 3 × 16 = 48.

(iii) If we take 2 2 2 for $p\ q\ r$ there are only 7 choices for $s\ t\ u$ as we are essentially back to the unordered case.

So finally the number of factorisations is 84 + 48 + 7 = 139.