2011 Senior Set 1 solutions

S1. The rectangular floor of a room is completely tiled with whole tiles, each of which is 15 cm square. The black tiles form a border of width one tile round the room. Within the black border all the tiles are red. There are exactly twice as many red tiles as black ones. Determine the possible lengths and corresponding widths for the room.

Solution

Let the number of tiles along the length of the room be m.

Let the number of tiles across the width of the room be n.

Then there are $2\left[(m-1)+(n-1)\right]$ black tiles in the border and $(m-2)(n-2)$ red tiles in the centre.

We require

$$\begin{aligned}(m-2)(n-2) &= 2\times 2\left[(m-1)+(n-1)\right]\\ mn-2m-2n+4 &= 4m+4n-8\\ mn-6m-6n+12 &= 0\\ (m-6)(n-6)-36+12 &= 0\\ (m-6)(n-6) &= 24\end{aligned}$$

The possible values for $m-6$ and $n-6$ are given in the table. (Note that $m>n$.)

$m-6$	$n-6$	m	n	length metres	width metres
24	1	30	7	4.5	1.05
12	2	18	8	2.7	1.2
8	3	14	9	2.1	1.35
6	4	12	10	1.8	1.5

There are 4 possible room dimensions, as given in the table above.

(The room is more of a corridor or a store cupboard!)

S2. Show that there is only one set of different positive integers, x, y, z, such that

$$1 = \frac{1}{x} + \frac{1}{y} + \frac{1}{z}$$

i.e. 1 can be expressed as the sum of the reciprocals of three different positive integers in only one way.

Deduce that, if n is any odd integer greater than 3, then 1 can be expressed as the sum of n reciprocals of different positive integers.

For which even integers is this possible? **Justify your answer**.

Solution

Since the integers must be different, let $x < y < z$.
x cannot be 1 because the left-hand side value 1 would then be reached.
If x is 2, then y must be 3 or more. If y is 3, then

$$\frac{1}{z} = 1 - \frac{1}{2} - \frac{1}{3} = \frac{1}{6}$$

and so one solution is

$$1 = \frac{1}{2} + \frac{1}{3} + \frac{1}{6}.$$

If $y = 4$, then z must be 5 or more, and the total of 1 would never be reached.

If x is 3, then y must be at least 4 and z at least 5. Hence again the total of 1 would never be reached.

If x is greater than 3, then y must be greater than 4 and z greater than 5, so that the shortfall would be even greater.
Thus there is only the one solution already found.

Moving to odd numbers of terms in excess of 3.

$$\begin{aligned} 1 &= \frac{1}{2} + \frac{1}{3} + \frac{1}{6}\left(\frac{1}{2} + \frac{1}{3} + \frac{1}{6}\right) \\ &= \frac{1}{2} + \frac{1}{3} + \frac{1}{12} + \frac{1}{18} + \frac{1}{36} \end{aligned}$$

Multiplying the last and smallest reciprocal by $1 = \frac{1}{2} + \frac{1}{3} + \frac{1}{6}$ has increased the number of reciprocals by 2, and this can be done repeatedly to obtain any odd number of different reciprocals with sum 1.

For just 2 reciprocals, let $1 = \frac{1}{x} + \frac{1}{y}$ with $x \leqslant y$.
If $x = 2$, then y must also equal 2. But then x and y are not different.
If x is 3 or more then y must be greater than or equal to 3 and the total can never reach 1.
Hence there is no solution for 2 reciprocals.

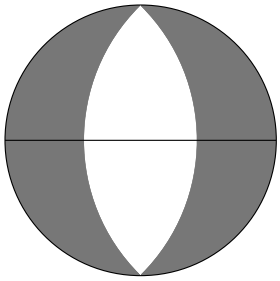
For 4 reciprocals

$$\begin{aligned} 1 &= \frac{1}{2} + \frac{1}{2}\left(\frac{1}{2} + \frac{1}{3} + \frac{1}{6}\right) \\ &= \frac{1}{2} + \frac{1}{4} + \frac{1}{6} + \frac{1}{12}. \end{aligned}$$

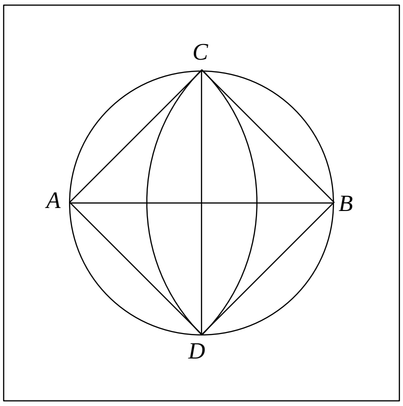
This satisfies the criteria and the number of reciprocals can be increased by 2 by multiplying the last and smallest reciprocal by $1 = \frac{1}{2} + \frac{1}{3} + \frac{1}{6}$ as before to obtain any even number of different reciprocals with sum 1.

Thus the only number of terms which cannot be used is 2.

S3. A circle has radius 1cm and AB is a diameter. Two circular arcs of equal radius are drawn with centres A and B. These arcs meet on the circle as shown. Calculate the shaded area.



Solution



$$BC^2 = 1^2 + 1^2 = 2$$

$$\angle CBA = 45° \text{ so } \angle CAD = 90°$$

$$\text{the radius of the sector } CBD \text{ is } BC = \sqrt{2}$$

$$\text{so the area of sector } CBD = \frac{1}{4}\pi BC^2 = \frac{\pi}{2}$$

$$CD = 2 \text{ so the area of } \Delta BCD = \frac{1}{2} \times 2 \times 1 = 1$$

$$\therefore \text{ the unshaded area} = 2\left(\frac{\pi}{2} - 1\right) = \pi - 2$$

$$\text{But the area of the full circle} = \pi \times 1^2 = \pi.$$

$$\text{So the shaded area} = \pi - (\pi - 2) = 2\,\text{cm}^2.$$

Alternative

Let C and D be the points shown in the diagram.

$AC^2 = 1^2 + 1^2 = 2$ so that the radius of the circular arc with centre A is $\sqrt{2}$.

Then the shaded area = (area of circle of radius 1) − 2(area of the segment of a circle, radius, $\sqrt{2}$, subtended by an angle of $\pi / 2$)

$= \pi - 2$(area of a sector of a circle radius , $\sqrt{2}$, subtended by an angle of $\pi / 2$ − area of ΔACD)

$= \pi - 2$(quarter of area of circle radius $\sqrt{2}$ − area of ΔACD)

$= \pi - 2\left(\tfrac{1}{4} \times 2\pi - \tfrac{1}{2} \times \sqrt{2} \times \sqrt{2}\right)$

$= \pi - 2\left(\tfrac{\pi}{2} - 1\right) = 2\text{ cm}^2$

S4. The thinking power of a multi-headed dragon depends on how many heads it has. The thinking power of a 'weyr' of dragons is the product of the number of heads on the individual dragons. A particular weyr has 100 heads available, how many dragons with what number(s) of heads will maximise the thinking power of the group?

Solution

1-headed dragons do not add to the power of a weyr and so are useless.

Let h be the number of heads on a dragon.

If $h > 4$, one dragon with h heads has power h whereas 2 dragons with 2 and $h - 2$ heads have power $2(h - 2) = h + (h - 4) > h$, so this is always worth doing.

$h = 4$: a 4-headed dragon has the same thinking power as 2 2-headed dragons.

Hence we only need consider dragons with 2 or 3 heads: for a group of 6 heads available, we could have either

3 2-headed dragons, with thinking power $2^3 = 8$

or 2 3-headed dragons, with thinking power $3^2 = 9$.

Clearly 2 3-headed dragons are more effective.

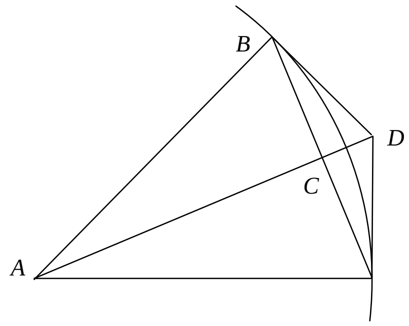
For the 100 head heads available, 96 of these should be used by 32 3-headed dragons. The remaining 4 heads could then be used by 2 2-headed dragons to optimise the thinking power at $3^{32} \times 2^2$

So 32 3-headed dragons and either 2 2-headed dragons or 1 4 headed dragon should be recruited.

S5. Show that the ratio of the area of a regular n-gon inscribed in a circle to the area of a regular n-gon circumscribing the same circle is $\cos^2 \frac{\pi}{n} : 1$.

Solution

Let the regular n-gon circumscribing the circle have its edges tangent to the circle at the vertices of the inscribed regular n-gon. Then one segment will look as shown below:



The ratio of the area of the inscribed n-gon to the area of the circumscribed n-gon is then the ratio of the area of ΔABC: area of ΔABD.

Note that $\angle ACB$ is a right angle as is $\angle ABD$.

Now $\angle BAC = \frac{1}{2} \cdot \frac{2\pi}{n} = \frac{\pi}{n}$. (Use of degrees would be acceptable.)

So the area of $\Delta ABC = \tfrac{1}{2} AC \times BC = \tfrac{1}{2} AB \cos\angle BAC \times AB \sin\angle BAC$.

The area of $\Delta ABD = \tfrac{1}{2} AB \times BD = \tfrac{1}{2} AB \times AB \tan\angle BAC$.

So $\frac{\text{area of } \Delta ABC}{\text{area of } \Delta ABD} = \frac{\sin\angle BAC \times \cos\angle BAC}{\tan\angle BAC} = \cos^2\angle BAC$,

$$\text{i.e. area of } \Delta ABC : \text{area of } \Delta ABD = \cos^2 \frac{\pi}{n}.$$