2010 Senior Set 2 solutions

S1. The shape of a fifty-pence piece is based on a regular heptagon which is a 7-sided polygon. The distance between each vertex and each of its two 'nearly opposite' vertices is 1 unit. The perimeter of the coin is formed by circular arcs of radius 1 unit which are centred on each vertex, and join the two nearly opposite vertices. Find the length of the perimeter of the coin.

Solution

The arc between two neighbouring vertices subtends an angle of $\frac{1}{7} \times 360°$ at the centre of the circumscribing circle.

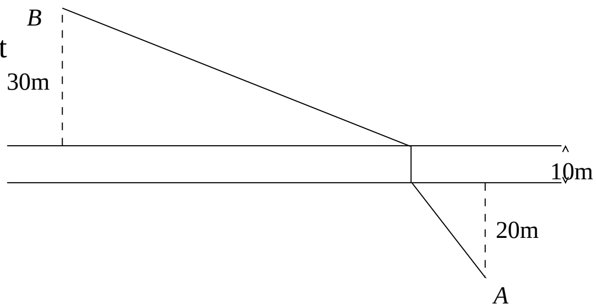
The inscribed angle is half of the central angle (the angle at the centre of a circle is twice the angle at the circumference), so the arc between two neighbouring vertices subtends an angle of $\frac{1}{7} \times 180°$ at the opposite vertex.

Thus the arc length between a neighbouring pair of vertices is $\frac{1}{7} \times$ circumference of a circle of radius $\frac{1}{2} = \frac{1}{7} \times \left(2\pi \times \frac{1}{2}\right) = \frac{1}{7}\pi$.

There are 7 arcs, so the total perimeter is $7 \times \frac{1}{7}\pi = \pi$.

The perimeter of the coin is π.

S2. A rabbit's burrow is at *A* and he knows that there are carrots in a garden at *B*, across a road, which is 10m wide. The burrow is 20m from the nearer edge of the road and the carrots are 30m beyond the other edge as shown in the diagram. The straight line distance from *A* to *B* is 80m.



The rabbit is wary of crossing the road and knows from past experience that he must cross directly across the road, not askew. What is the length of the shortest possible route for the rabbit from the burrow to the carrots?

Solution

Since the rabbit always crosses straight across the road, any route he takes will be 10m plus the distance from *A* to a point on one edge of the road plus the distance from the opposite point on the road to *B*. But this amounts to removing the road and considering the shortest distance then from *A* to *B*, which is, of course, a straight line.

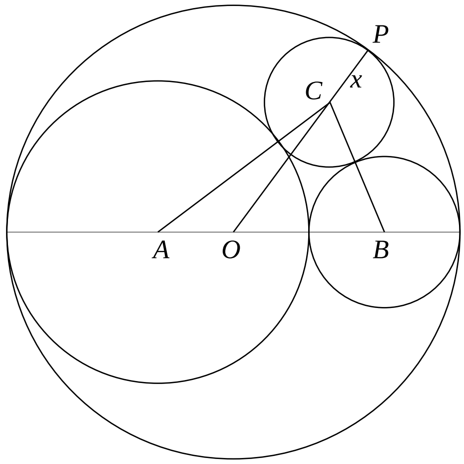
To obtain this we use Pythagoras theorem. On the original diagram, it is 80m from *A* to *B*. So the 'horizontal' distance between *A* and *B* is $\sqrt{80^2 - 60^2}$.

With the road 'removed', the new distance between *A* and *B* is $\sqrt{50^2 + (80^2 - 60^2)} = 10\sqrt{53}$ metres.

So the shortest distance the rabbit has to travel is $10 + 10\sqrt{53}$ metres.

S3. One disc of 20 cm diameter and one of 10 cm diameter are cut from a disc of plywood of diameter 30 cm. What is the diameter of the largest disc that can be cut from the wood that remains? (Ignore the thickness of the saw cut.)

Solution



Let A be the centre of the disc of radius 10 cm. Let B be the centre of the disc of radius 5 cm. Let O be the centre of the disc of radius 15 cm. Let C be the centre of the disc to be determined and let its radius be x cm. Note that OC extended to the point of tangency P of the disc centre O and the outer disc is a radius of the large disc.

We then have $AC = 10 + x$, $BC = 5 + x$, $OC = 15 - x$. Also, $AO = 15 - 10 = 5$ and $OB = 15 - 5 = 10$.
So, from triangle AOC we have

$$(10 + x)^2 = (15 - x)^2 + 5^2 - 2(15 - x) \times 5 \cos\angle AOC$$

$$(5 + x)^2 = (15 - x)^2 + 10^2 - 2(15 - x) \times 10 \cos\angle BOC$$

Since $\angle BOC + \angle AOC = 180°$, $\cos\angle BOC = -\cos\angle AOC$, so we add twice the first equation to the second to get:

$$2(10 + x)^2 + (5 + x)^2 = 3(15 - x)^2 + 50 + 100$$

$$200 + 40x + 2x^2 + 25 + 10x + x = 675 - 90x + 3x^2 + 150.$$

Solving gives $140x = 600$ so the disc has diameter $8\frac{4}{7}$ cm.

S4. Calculate

$$67^2$$
$$667^2$$
$$6667^2$$
$$66667^2$$

Find the value of the square of the number consisting of one million sixes, followed by one seven. Justify your answer.

Solution

$$67^2 = 4489$$
$$667^2 = 44889$$
$$6667^2 = 44448889$$
$$66667^2 = 4444488889$$

This suggests that for $666\ldots 67^2$, where there are N sixes, the answer will be a number containing $N + 1$ fours, N eights and a single nine.

Now

$$\begin{aligned}\underbrace{666\ldots}_{N}67 = \underbrace{666\ldots}_{N}66 + 1 &= \frac{6}{9}\Big(\underbrace{999\ldots 99}_{N}\Big) + 1\\ &= \frac{2}{3}\left(10^{N+1} - 1\right) + 1\\ &= \frac{2}{3}10^{N+1} + \frac{1}{3}.\end{aligned}$$

Hence

$$\begin{aligned}\Big(\underbrace{666\ldots\ 67}_{N}\Big)^2 &= \left(\frac{2}{3}10^{N+1} + \frac{1}{3}\right)^2 = \frac{4}{9}10^{N+2} + \frac{4}{9}10^{N+1} + \frac{1}{9}\\ &= \frac{4}{9}\Big(\underbrace{999\ldots\ 99}_{2N+2} + 1\Big) + \frac{4}{9}\Big(\underbrace{999\ldots\ 99}_{N+1} + 1\Big) + \frac{1}{9}\\ &= \underbrace{444\ldots\ 44}_{2N+2} + \frac{4}{9} + \underbrace{444\ldots\ 44}_{N+1} + \frac{4}{9} + \frac{1}{9}\\ &= \underbrace{444\ldots\ 44}_{N+1}\underbrace{888\ldots\ 88}_{N+1} + 1\\ &= \underbrace{444\ldots\ 44}_{N+1}\underbrace{888\ldots\ 8}_{N}9\end{aligned}$$

So value of the square of the number consisting of one million sixes, followed by one seven is the number which starts with 1 000 001 fours, then 1 000 000 eights and finally a nine.

S5. In a wood there are more than 100 trees and all the trees have leaves on them. The number of trees in the wood is more than double the number of leaves on any one tree in the wood. Identify which of the following statements must be true:

- at least two trees have the same number of leaves on them;
- at least three trees have the same number of leaves on them;
- at least four trees have the same number of leaves on them.

Explain your answer in each case.

Solution

Let the maximum number of leaves on any tree be m and the number of trees in the wood be n. So $2m < n$.

1. True. The number of leaves a tree can have is $1,2,3,\ldots, m$. So if all the trees had a different number of leaves there could be at most m trees.

2. True. For each number of leaves, if there were just two trees with that number of leaves, there could be at most $2m$ trees. Since $2m < n$ this is false so there must be at least three trees with the same number of leaves.

3. Not necessarily true. It suffices to give an example. Let $n = 120$ and $m = 50$. There could be 30 pairs of trees having $1,2,3,\ldots,30$ leaves on them and 20 triples of trees having $31,32,33,\ldots,50$ leaves on them. This gives 120 trees, none with more than 50 leaves and no four trees have the same number of leaves.