2010 Senior Set 1 solutions

S1. Katie had a collection of red, green and blue beads. She noticed that the number of beads of each colour was a prime number and that the numbers were all different. She also observed that if she multiplied the number of red beads by the total number of red and green beads she obtained a number exactly 120 greater than the number of blue beads. How many beads of each colour did she have?

Solution

Suppose Katie had r red beads, g green beads and b blue beads. Then

$$r(r + g) = 120 + b.$$

If $b = 2$, then the right-hand side is $122 = 2 \times 61$ and so $r = 2$. But the numbers of beads are all different. So b must be an odd prime.

This means that the right-hand side is odd. Thus both r and $r + g$ must be odd, so g must be even and prime.

So $g = 2$ and the equation becomes $r^2 + 2r = 120 + b$.

Hence $b = r^2 + 2r - 120 = (r - 10)(r + 12)$.8/5/16

Since b is prime, $r - 10 = 1$ so $r = 11$ and $b = 23$.

Thus Katie had 11 red beads, 2 green beads and 23 blue beads.

S2. Ant and Dec had a race up a hill and back down by the same route. It was 3 miles from the start to the top of the hill. Ant got there first but was so exhausted that he had to rest for 15 minutes. While he was resting, Dec arrived and went straight back down again. Ant eventually passed Dec on the way down just half a mile before the finish.

Both ran at a steady speed uphill and downhill and, for both of them, their downhill speed was one and a half times faster than their uphill speed. Ant had bet Dec that he would beat him by at least a minute.

Did Ant win his bet?

Explain your answer.

Solution

Let Ant's uphill speed be a mph and Dec's be b mph.

Suppose that Ant had been resting for x hours when Dec arrived (where x is between 0 and $\frac{1}{4}$). Then, calculating their times to the top of the hill and then until Ant passed Dec on the way down we have:

Time going up $\quad \frac{3}{a} + x = \frac{3}{d},$

and the time going down $\quad \frac{\frac{5}{2}}{\frac{3}{2}a} + \left(\frac{1}{4} - x\right) = \frac{\frac{5}{2}}{\frac{3}{2}d}.$

So rearranging each of these:

$$\frac{1}{d} - \frac{1}{a} = \frac{x}{3} \text{ and } \frac{1}{d} - \frac{1}{a} = \frac{3}{5}\left(\frac{1}{4} - x\right).$$

$$\frac{x}{3} = \frac{3}{5}\left(\frac{1}{4} - x\right)$$

$$\frac{5x}{9} = \frac{1}{4} - x$$

$$20x = 9 - 36x \Rightarrow x = \frac{9}{56}.$$

Dec's time for the whole race minus Ant's time for the whole race =

$$\left(\frac{3}{d} + \frac{3}{\frac{3}{2}d}\right) - \left(\frac{3}{a} + \frac{1}{4} + \frac{3}{\frac{3}{2}a}\right) = 5\left(\frac{1}{d} - \frac{1}{a}\right) - \frac{1}{4} = \frac{1}{5} \times \frac{9}{56} - \frac{1}{4} = \frac{1}{56} > \frac{1}{60}.$$

So Ant did beat Dec by more than a minute and won his bet.

S3. Two numbers contain the same digits in a different order. Explain why the difference between the numbers is always a multiple of 9.

Solution

Let the original number be N where

$$N = a_n \times 10^n + a_{n-1} \times 10^{n-1} + \dots + a_1 \times 10 + a_0.$$

Then the second number with digits in a different order, N', will be

$$N' = a_n \times 10^{i_n} + a_{n-1} \times 10^{i_{n-1}} + \dots + a_1 \times 10^{i_1} + a_0 \times 10^{i_0}$$

where $i_0, i_1, \dots , i_n$ is a rearrangement of the digits $0, 1, \dots , n$.

If we subtract to give $N - N'$ and then consider the term in, say, a_j, their difference is

$$a_j \left(10^j - 10^{i_j}\right) \text{ if } 10^j > 10^{i_j},$$
$$a_j \left(10^{i_j} - 10^j\right) \text{ if } 10^{i_j} > 10^j.$$

Extracting the lower power of 10 from the bracket gives an expression of the form: $a^j 10^p \left(10^q - 1\right)$, for some p and q.

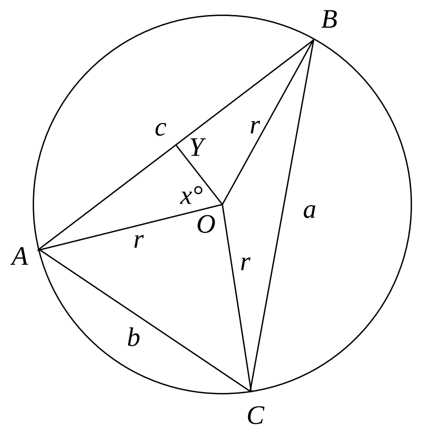
For all q, $10^q - 1$ is divisible by 9.

This is true for all j and so, $N - N'$ is always a multiple of 9.

S4. Let ABC be an acute-angled triangle with sides of lengths a, b, c and area X. Show that the radius of the circle through A, B and C is $\dfrac{abc}{4X}$.

Solutions

Method 1



Using convential notation we let $AB = c$, etc.
From a standard result, we can say

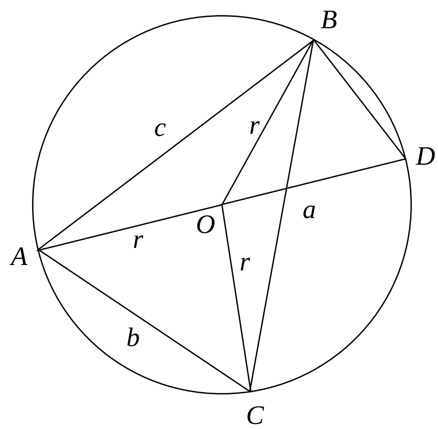
$$X = \tfrac{1}{2}ab \sin \angle C.$$

Since $AO = OB$, ΔAOB is isosceles and as $AY = YB$, OY is perpendicular to AB and $\angle AOB = 2x°$. But, considering the chord AB, $\angle AOB = 2\angle ACB$ therefore $x° = \angle C$. So from ΔAOY

$$\sin C = \sin x° = \frac{\tfrac{1}{2}c}{r} = \frac{c}{2r}$$

hence $X = \dfrac{ab}{2}\dfrac{c}{2r} = \dfrac{abc}{4r} \Rightarrow r = \dfrac{abc}{4X}.$

Method 2



Using convential notation we let $AB = c$, etc.
From a standard result, we can say

$$X = \tfrac{1}{2}ab \sin \angle C.$$

Now draw in the diameter AD. We therefore have two results which are useful: $\angle D = \angle C$ since they are angles in the same segment; and, $\angle ABD = 90°$ as it is the angle in a semi-circle. These give

$$\sin \angle C = \sin \angle D = \frac{c}{2r},$$

hence $X = \dfrac{ab}{2}\dfrac{c}{2r} = \dfrac{abc}{4r} \Rightarrow r = \dfrac{abc}{4X}.$

Method 3

Using convential notation we let $AB = c$, etc.
From a standard result, we can say

$$X = \tfrac{1}{2}ab \sin \angle C.$$

We now quote the full result of the well-known Sine Rule:

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} = 2r$$

$$\Rightarrow \sin \angle C = \frac{c}{2r}$$

hence

$$X = \frac{ab}{2}\frac{c}{2r} = \frac{abc}{4r} \Rightarrow r = \frac{abc}{4X}.$$

S5. John said "I am told that there is only one number between 2 and 200 000 000 000 000 which is a perfect square, a perfect cube and also a perfect fifth power. I am sure there must be more than one, but I have looked at all the numbers up to 100 000 and I haven't found any! I am getting fed up doing this."

Was the information John given correct? **Explain your answer.**

Solution

Consider the integers A, B, C, D and E. Suppose $A = B^2 = C^3 = D^5$.

Now C^2 divides $C^3 = B^2$ and so C divides B .

$(B/C)^6 = B^6/C^6 = A^3/A^2 = A$. So $A = D^5 = E^6$.

Now E^5 divides $E^6 = D^5$ so that E divides D.

$(D/E)^{30} = D^{30}/E^{30} = A^6/A^5 = A$.

So A is an exact 30th power. Now $3^{30} > 2 \times 10^{14}$ and $2^{30} = 1\,073\,741\,824$ so this must be the required number and there is only one.