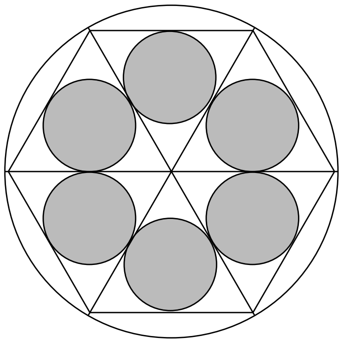
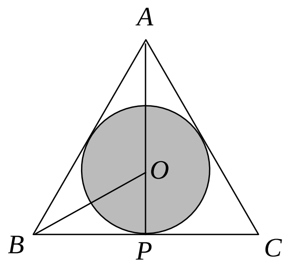
2009 Senior Set 2 solutions

S1. The diagram shows a regular hexagon with its diagonals drawn and six circles fitted into the regions created. What fraction of the circumscribing circle is shaded?



Solution

Consider one of the shaded circles.



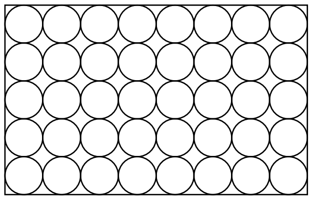
Let $OP = r$ = radius of the small circle and $AB = BC = CA = R$ = radius of the large circle. Then $AB^2 = AP^2 + BP^2$. So $R^2 = AP^2 + \left(\frac{1}{2}R\right)^2$. Thus $AP^2 = \frac{3}{4}R^2$. Since $OB = OA$, $AP = r + OA = r + OB$, i.e. $OB = AP - r$. Now, applying Pythagoras' Theorem to $\triangle OBP$, we get

$$\begin{aligned} OB^2 &= BP^2 + OP^2 \\ (AP - r)^2 &= \left(\tfrac{1}{2}R\right)^2 + r^2 \\ AP^2 - 2AP.r + r^2 &= \tfrac{1}{4}R^2 + r^2 \\ \tfrac{3}{4}R^2 - \tfrac{2\sqrt{3}}{2}Rr &= \tfrac{1}{4}R^2 \\ \tfrac{1}{2}R^2 &= \sqrt{3}Rr \\ R &= 2\sqrt{3}\,r \end{aligned}$$

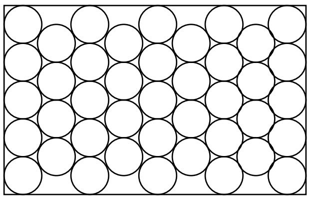
Hence, the area of the circumcircle is $\pi R^2 = 12\pi r^2$ which is exactly twice the shaded area.

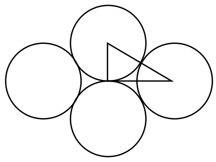
Alternative Solution. Area of circumscribed circle is πR^2. Shaded area is $6\pi r^2$ where $r = OP$. From the triangle OBP, $BP = \frac{1}{2}BC = \frac{1}{2}AB = \frac{1}{2}R$. So $r / \left(\frac{1}{2}R\right) = \tan 30° = 1/\sqrt{3}$. So $r = \dfrac{R}{2\sqrt{3}}$.
So shaded area is $6 \times \pi r^2 = 6\pi\dfrac{R^2}{12} = \dfrac{1}{2}\pi R^2$.

S2. Forty cylindrical tubes, each with diameter one inch and equal lengths, are packed as shown snugly in 5 rows of 8 each in a box so that they may be transported without rattling. Show that the box could be repacked with forty-one of the same sized cylindrical tubes. Will they now rattle?



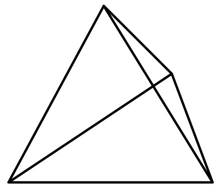
Solution

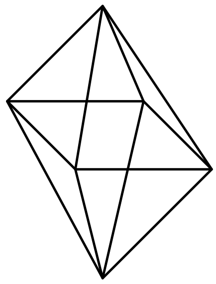




The box can be repacked as shown.
Consider a cluster of four circles. The right-angle triangle shown has known sides $\frac{1}{2}$, 1. Hence by Pythagoras' Theorem, the third side is $\sqrt{1^2 - \frac{1}{2}^2} = \sqrt{\frac{3}{4}} = \frac{1}{2}\sqrt{3}$.
This represents the distance between the centres of adjacent columns of which there are 9. Hence the total distance from left to right is $\frac{1}{2} + \frac{8}{2}\sqrt{3} + \frac{1}{2} = 4\sqrt{3} + 1$ and this is less than 8.
Thus they can be packed in the box.
Since $4\sqrt{3} + 1$ is strictly less than 8 there will be room for the tubes to rattle in the box.

S3. A regular tetrahedron and a regular octahedron have edges of the same length. Find the ratio of their volumes.

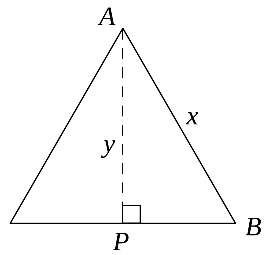




Solution 1

The problem can be solved by using the standard formula that the volume of a pyramid is equal to one third of the area of its base times its height. Let the edges have length x units.

(a) For the tetrahedron:

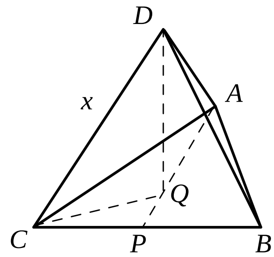


The diagram shows the base of the tetrahedron. Since AP is an axis of symmetry $PB = \frac{1}{2}x$ and its altitude, AP, can be calulated by Pythagoras' theorem:

$$y^2 = x^2 - \left(\tfrac{1}{2}x\right)^2 = \tfrac{3}{4}x^2 \Rightarrow y = \tfrac{\sqrt{3}}{2}x.$$

Hence the area of the base is $\frac{1}{2}x\left(\frac{\sqrt{3}}{2}x\right) = \frac{\sqrt{3}}{4}x^2$.

We now need to calculate the height of the tetrahedron.



CQ bisects $\angle ACB$ so $\angle QCP = 30°$ and we can use some trigonometry

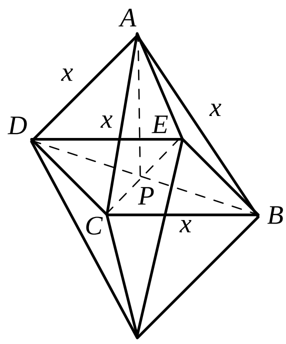
$$\cos 30° = \frac{CP}{CQ} \Rightarrow CQ = \frac{\frac{1}{2}x}{\frac{\sqrt{3}}{2}} = \frac{x}{\sqrt{3}}$$

and now use Pythagoras' theorem again:

$$DQ^2 = x^2 - \left(\frac{x}{\sqrt{3}}\right)^2 = \frac{2}{3}x^2.$$

Thus the volume of the tetrahedron is $\frac{1}{3} \times \frac{\sqrt{3}}{4}x^2 \times \sqrt{\frac{2}{3}}x = \frac{\sqrt{2}\,x^3}{12}$.

(a) For the octahedron:



An octahedron can be considered as a pair of square-based pyramids.

The area of the base $BCDE$ is x^2.

The height is AP which is equal to each of CP, EP, DP, PB and so half of BD (and CE).

$$CE^2 = x^2 + x^2 \Rightarrow CE = \sqrt{2}x \Rightarrow AP = \tfrac{1}{2}\sqrt{2}x.$$

Hence, the volume of the octahedron is

$$2 \times \left(\frac{1}{3} \times x^2 \times \frac{\sqrt{2}}{2}x\right) = \frac{\sqrt{2}}{3}x^3 = 4 \times \left(\frac{\sqrt{2}\,x^3}{12}\right).$$

Thus the volume of the octahedron is is four times the volume of the tetrahedron.

Solution 2

If one vertex of the regular tetrahedron is truncated by a plane passing through the midpoints of the edges from that vertex, a smaller regular tetrahedron is cut off whose edges are one half the length of the edges of the original tetrahedron. Thus the volume of this smaller tetrahedron is 1/8 of the volume of the larger tetrahedron.

If the truncation is carried out on all four vertices, the solid left is a regular octahedron whose edge lengths are half those of the original tetrahedron.

Thus this octahedron has volume 1/2 of the volume of the original tetrahedron. Also the edge lengths of this octahedron are the same as the edge lengths of the smaller tetrahedron. Since the volume of the smaller tetrahedron is 1/8 of the volume of the original tetrahedron, the ratio of the volume of a regular tetrahedron to the volume of a regular octahedron where they both have the same edge lengths is 1:4.

S4. A stall at a Farmers' Market sells chocolate truffles, tablet and Turkish delight. Each is sold per piece and the price of each is different. A customer purchased as many pieces of each sweet as its price per sweet in pence and paid an average of 7p per sweet.

Truffles cost the most, tablet costs less and Turkish delight is the cheapest. Turkish delight costs at least 4p per piece. One of tablet or Turkish delight costs 3p less than a truffle.

The customer spent less than £1.50. How many of each kind of sweet did he buy?

Solution

Let the numbers of each of truffles, tablet and Turkish delight bought be c, t and d. Then the total cost is

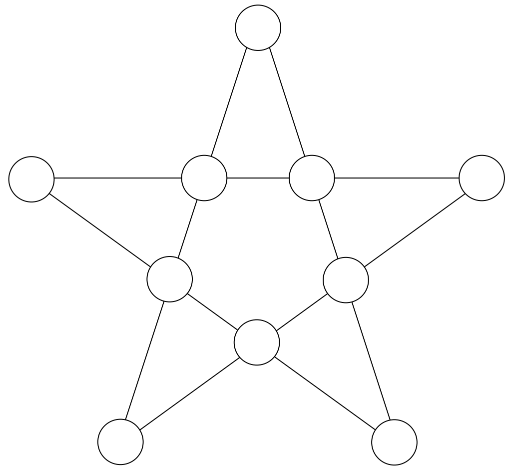
$$c^2 + t^2 + d^2 = 7(c + t + d) \qquad (1)$$

Since truffles cost most, $c > 7$; if $c = 8$, then either t or d is 5. Assume that $t = 5$, and substitute into (1). This gives $89 + d^2 = 91 + 7d \Rightarrow d^2 - 7d - 2 = 0$. This does not give a solution. So $c \neq 8$.

If $c = 9$ and $t = 6$, then $d^2 - 7d + 12 = 0$ with solutions $d = 3$ and $d = 4$. Since $d \geqslant 4$, this gives $c = 9$, $t = 6$ and $d = 4$.

This is the only possible solution since, if $c \geqslant 10$, then $t \geqslant 7$ and $d \geqslant 4$ giving a total cost greater than £1.50.

S5. The diagram shows a Magic Star, which is similar to a Magic Square, the numbers 1 to 10 have to be placed in the circles so that the sum of the numbers on each line is the same. Prove that this cannot be done.



Solution

Let N be the total on each line. There are 5 lines giving a total of $5N$.

But each circle lies on two lines so in that total we have counted each number twice. Thus $5N = 2(1 + 2 + \ldots +10) = 110$. So $N = 22$.

Note that each pair of lines has just one number in common and each number lies on exactly two lines.

List the possibilities starting with those with the highest numbers. These are

(10, 9, 2, 1), (10, 8, 3, 1), (10, 7, 4, 1), (10, 7, 3, 2), (10, 6, 5, 1), (10, 6, 4, 2), (10, 5, 4, 3),
(9, 8, 4, 1), (9, 8, 3, 2), (9, 7, 5, 1), (9, 7, 4, 2), (9, 6, 5, 2), (9, 6, 4, 3),
(8, 7, 6, 1), (8, 7, 5, 2), (8, 7, 4, 3), (8, 6, 5, 3), (7, 6, 5, 4).

The two lines through 10 can only have that number in common so we can pair them off:

(10, 9, 2, 1), (10, 5, 4, 3) (10, 8, 3, 1), (10, 6, 4, 2) (10, 7, 3, 2), (10, 6, 5, 1)

Note that (10, 7, 4, 1) has no partner.

Suppose the pair is (10, 9, 2, 1) and (10, 5, 4, 3). The other line through 9 must contain exactly one of 5, 4, 3 and none of 10, 2, 1. By checking the list above there is no such line.

Suppose the pair is (10, 7, 3, 2) and (10, 6, 5, 1). The other line through 7 must contain exactly one of 6, 5, 1 and none of 10, 3, 2. Again there is no such line.

Suppose the pair is (10, 8, 3, 1) and (10, 6, 4, 2). The other line through 8 must contain exactly one of 6, 4, 2 and none of 10, 3, 1. There is one possibility (8, 7, 5, 2). We have now used up all the numbers except 9. The remaining two lines must contain 9 and 7 and one from each of lines 1 and 2 (those containing 10) and 9 and 5 and one from each of lines 1 and 2. Again from the list above there is no such line.

Thus no magic star can contain the number 10 and so no magic star exists.