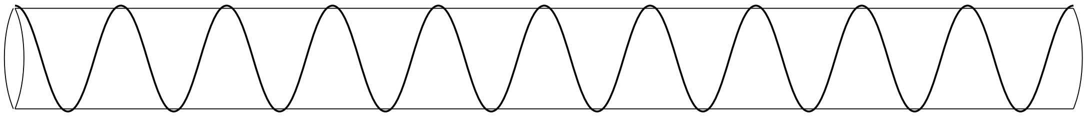
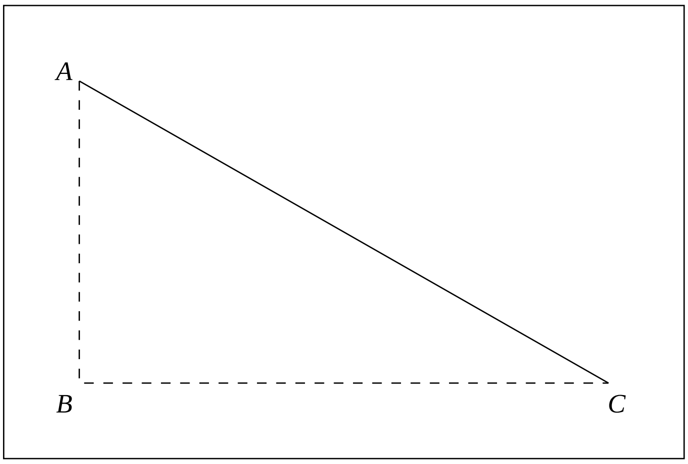
2009 Senior Set 1 solutions

S1



Ten turns of a wire are helically wrapped round a cylindrical tube with outside circumference 4 inches and length 9 inches. At the start and the finish, the end of the wire is at the top. Find the length of the wire.

Solution 1
Consider very carefully fixing one end of the wire on a board and rolling the cylinder until all the wire is on the board.



$BC = 9$ as it is the length of the cylinder. AB is equal to 10 revolutions of the cylinder. Hence

$$AB = 10 \times 4 = 40$$

So, using Pythagoras theorem,

$$\begin{aligned} AC^2 &= AB^2 + BC^2 \\ &= 1600 + 81 = 41^2 \end{aligned}$$

Thus the length of the wire is 41 inches.

Solution 2
The length of the wire will be 10 times as long as the piece of wire wrapped once round a cylindrical tube of circumference 4 inches and length 9/10 inches. Cut this open along the cylindrical element to obtain a rectangle of width 4 inches and height 9/10 inches. The wire is then a diagonal of this rectangle so, by Pythagoras has length in inches $\sqrt{4^2 + \left(\frac{9}{10}\right)^2} = \frac{41}{10}$.
Thus the length of the wire is 41 inches.

S2 A sequence of numbers $\{a_n\}$ starts as follows:

$$1, 3, 4, 7, 11,18, 29, 47, 76, 123,\ldots .$$

So $a_1 = 1$, $a_2 = 3$, $a_3 = 4$, etc

Identify numbers in the sequence which are in the sequence are

(a) multiples of 3? (c) multiples of 7?

(b) multiples of 5? (d) multiples of 21?

Solution

Notice that $a_{n+2} = a_{n+1} + a_n$.

First consider the remainders on dividing by 3:

$$1,\ 0,\ 1,\ 1,\ 2,\ 0,\ 2,\ 2,\ 1,\ 0,\ 1,\ 1,\ \ldots$$

As soon as two consecutive remainders are the same as two earlier in the sequence, then all subsequent ones will be repeated. So every number a_{4n+2} is a multiple of 3 for any integer $n \geqslant 0$.

Consider the remainders on dividing by 5:

$$1,\ 3,\ 4,\ 2,\ 1,\ 3,\ 4,\ 2,\ \ldots$$

Using the same principle, no number will be divisible by 5.

Consider the remainders on dividing by 7:

$$1,\ 3,\ 4,\ 0,\ 4,\ 4,\ 1,\ 5,\ 6,\ 4,\ 3,\ 0,\ 3,\ 3,\ 6,\ 2,\ 1,\ 3,\ \ldots$$

So the numbers a_{8n+4} are divisible by 7 for any integer $n \geqslant 0$.

If a number is to be divisible by 21, it will be divisible by both 3 and 7. Thus there must be integers n, m such that $4n + 2 = 8m + 4$. But that is impossible since $4n + 2$ is never divisible by 4 and $8m + 4$ is always divisible by 4. Hence there are no numbers divisible by 21.

S3 How many zeroes are there at the end of the number which is the product of the integers 1 to 200? Justify your answer.

Solution

The product can be written as

$$1 \times 2 \times 3 \times 4 \times 5 \times \ \dots \ \times 198 \times 199 \times 200.$$

From this we can see there are 40 numbers divisible by 5

$$5,\ 10,\ 15,\ 20,\ 25,\ \dots\ ,\ 190,\ 195,\ 200.$$

There are 8 numbers divisible by $5^2 = 25$

$$25,\ 50,\ 75,\ 100,\ 125,\ 150,\ 175,\ 200.$$

There is 1 number divisible by $5^3 = 125$

$$125.$$

From this we can see the product must be divisible by $5^{40} \times 5^8 \times 5^1 = 5^{49}$. As no other 5's are involved in any of the terms it cannot be divisible by any higher power of 5.

As there are more than 49 even values (and hence, factors of 2) in the product, it follows that it must be divisible by 2^{49}.

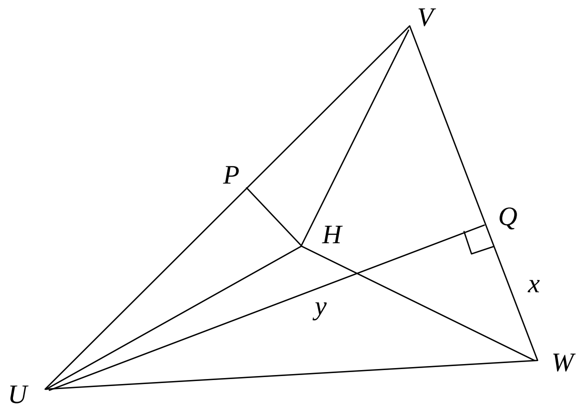
Combining these values the product must be divisible by $5^{49} \times 2^{49} = 10^{49}$.

The product can be written as $k \times 10^{49}$ where k is a positive integer not divisible by 5. This means that k cannot end in 0, and the number of zeroes at the end of the product must be 49.

S4 Sitting on the train I asked my travelling companion how far he had to go after he got off the train. He said “If I get off at Udnie, or if I continue to the next station at Vernon, another 15 miles down the line, I have the same distance to travel home. In fact, if I changed at Udnie and took the branch line to Waterhall, a further 13 miles by train, I would still have the same distance to travel home after I got off the train.” I know that it is 14 miles from Waterhall to Vernon. How far was my companion's home from the stations?

(Assume that all distances are ‘as the crow flies’.)

Solution



In the diagram U, V, W represent the stations and H the home. So in miles $UV = 15$, $UW = 13$ and $VW = 14$. Construct a perpendicular from U to VW at Q. Let $WQ = x$ and $UQ = y$. Then from triangles UQW and UVQ we have $y^2 + x^2 = 13^2$ and $y^2 + (14 - x)^2 = 15^2$. Subtracting these we get $14^2 - 28x = 15^2 - 13^2 = 56$. Solving this gives $x = 5$ and so $y = 12$. So $\sin\angle UVW = \frac{12}{15}$. Since U, V, W all lie on a circle, the angle subtended by UW at the centre H is twice the angle subtended at the circumference V. So $\angle UHW = 2\angle UVW$.

From the cosine rule in triangle UHW, we have

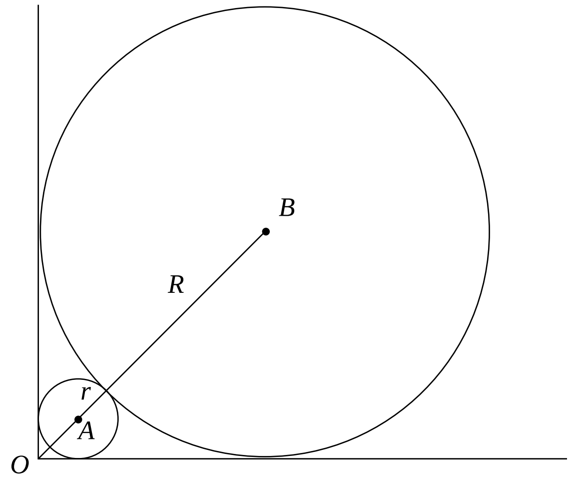
$13^2 = 2r^2 - 2r^2\cos\angle UHW = 2r^2(1 - \cos 2\angle UVW) = 4r^2\sin^2 UVW$. So

$13 = 2r\sin\angle UVW = 2r \times \frac{12}{15}$. So $r = 13 \times \frac{15}{2 \times 12} = \frac{65}{8}$.

So the distance $8\frac{1}{8}$ miles.

S5 In a two-dimensional space invaders game, all the players are circular. A large player is chasing a smaller one inside a rectangular arena. If two players touch, the smaller one is eaten. The smaller player stops right in a corner. What 'size' must it be in relation to the larger player if it is to survive?

Solution



Let the small player have centre *A* and radius *r* and the large one centre *B* and radius *R*. Then

$$\begin{aligned} OB &= R + r + OA \\ &= R + r + \sqrt{2}r \end{aligned}$$

But

$$\begin{aligned} OB &= \sqrt{2}R \\ \therefore \sqrt{2}R &= R + r + \sqrt{2}r \\ R(\sqrt{2} - 1) &= r(\sqrt{2} + 1) \\ \frac{r}{R} &= \frac{\sqrt{2} - 1}{\sqrt{2} + 1} \\ &= \left(\sqrt{2} - 1\right)^2 = 3 - 2\sqrt{2}. \end{aligned}$$

Hence the required ratio of the smaller player to the larger is $3 - 2\sqrt{2}$.

Alternative:

Project the lengths onto the horizontal line through *O*.

$$\begin{aligned} R &= r + \frac{r}{\sqrt{2}} + \frac{R}{\sqrt{2}} \\ R\sqrt{2} &= r\sqrt{2} + r + R \\ R(\sqrt{2} - 1) &= r(\sqrt{2} + 1) \\ \frac{r}{R} &= \frac{\sqrt{2} - 1}{\sqrt{2} + 1} \\ &= \left(\sqrt{2} - 1\right)^2 = 3 - 2\sqrt{2}. \end{aligned}$$