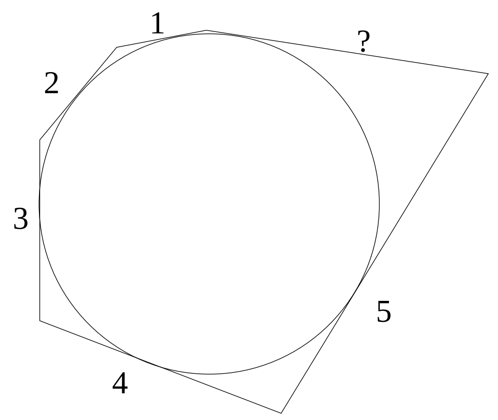
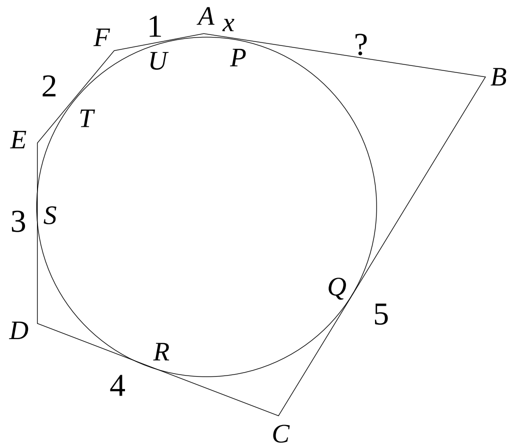
2008-2009 Senior Division Set 2 Solutions

S1. In the hexagon shown, all edges are tangent to the circle. If their lengths are 1, 2, 3, 4 and 5 as illustrated, what is the length of the remaining edge?



Solution

From each vertex of the hexagon, the two tangents have the same length. Each edge is made up of two such tangent lines.



Introduce letters A, B, C, D, E, F for the hexagon and P, Q, R, S, T, U for the points of contact. Also, let $AP = x$.

So

$$\begin{aligned} AU = x \Rightarrow\ & UF = 1 - x = FT \\ \Rightarrow\ & TE = 1 + x = ES \\ \Rightarrow\ & SD = 2 - x = DR \\ \Rightarrow\ & RC = 2 + x = CQ \\ \Rightarrow\ & QB = 3 - x = BP \end{aligned}$$

Therefore

$$AB = AP + PB = x + (3 - x) = 3.$$

Thus the length of the remaining edge is 3.

S2. An unusual team marathon is being held. Each team has two members who run and cycle over the course which is 42 km long. The rules are:

1. Each team starts together at the start line.
2. Each team is allowed only one bike and only one team member may ride the bike at any time. (This usually means that one member cycles at the start, puts the bike down at some point and then runs to the finish line. The other team member starts by running. When he or she reaches the bike, they pick it up and cycle to the finish line).
3. Both team members have to cross the finish line.
4. The time recorded for a team is the time for the second member to cross the finish line or the time of both members if they cross the finish line together. (This means that teams should work out before the race starts exactly where they should leave the bike to give the fastest recorded time).

Team	Member 1	Running	Cycling	Member 2	Running	Cycling
A	Zak	12	28	Sandra	12	28
B	Archie	16	35	Kirsty	10	15
C	Liam	14	25	Niamh	10	35

The table shows the steady running and cycling speeds, all in km/hr, for the three teams. Determine for each team, their best possible time for the race.

Solution

Team A. Since both team members run and cycle at the same speed, they should each cover the distance by running half the distance and cycling half the distance. So time taken will be $\frac{21}{12} + \frac{21}{28} = \frac{7}{4} + \frac{3}{4} = 2.5$ hours.

Team B. Since Archie's running and cycling speeds are both faster than both of Kirsty's speeds, Kirsty should cover the whole distance cycling and will be the second team member over the finish line. So time recorded will be $\frac{42}{15} = \frac{14}{5} = 2.8$ hours.

Team C. The point where the bike will be dropped will be chosen such that the two members cross the line together for the shortest recorded time. Suppose the bike is left after x km. Let Liam run first. Then he will take $\frac{x}{14} + \frac{42 - x}{25}$ hours and Niamh will take $\frac{x}{35} + \frac{42 - x}{10}$ hours. So we want to equate these two which gives

$$35 \times 10\left(25x + 14(42 - x)\right) = 14 \times 25\left(10x + 35(42 - x)\right).$$

Solving this we get $x = \frac{49}{2}$. So the time taken is $\frac{49}{2 \times 14} + \frac{42 - \frac{1}{2} \times 49}{25} = 2.45$ hours.

S3. Every polyhedron satisfies Euler's formula. This states that, if V is the number of vertices of the polyhedron, E the number of edges and F the number of faces, then $V - E + F = 2$.

A *rhombic triacontahedron* is a semi-regular polyhedron which has 30 congruent faces which are rhombuses and two types of vertices. At vertices of Type 1, three rhombuses meet and the angles there are all obtuse; at vertices of Type 2, five rhombuses meet and the angles are all acute. Use Euler's formula to help determine how many edges there are and how many vertices of each Type.

A new semi-regular polyhedron is formed by suitably truncating all the vertices of Type 2 of the rhombic triacontahedron described above. In this case all the vertices have the same number of edges meeting at them but there are two Types of faces. Describe this new polyhedron, giving the Types of faces, how many of each type there are and how many edges and vertices.

Solution

Each edge of the triacontahedron has a face on either side and each face has 4 edges. So in this case, $2E = 4F$. Thus there are 60 edges. Let V_1 be the number of vertices of Type 1 and V_2 the number of Type 2. By Euler's formula, $(V_1 + V_2) = 32$. At each vertex of Type 1, three faces meet and at each vertex of Type 2, five faces meet. Since each face has four vertices that means that $3V_1 + 5V_2 = 4F = 120$. Substitute $V_2 = 32 - V_1$ to get $3V_1 + 5(32 - V_1) = 120$. So $2V_1 = 40$. So $V_1 = 20$ and $V_2 = 12$.

For the second polyhedron, there is a new face for each vertex of Type 2 and still the same number of old faces. Thus there are 42 faces in all. The new faces are all pentagons and the old faces are now hexagons as they come from rhombuses with two opposite corners, with acute angles, cut off. Thus there are 12 pentagons and 30 hexagons. Each of the pentagons contributes 5 new vertices, but 12 vertices have been lost. So there are $20 + 60 = 80$ vertices. Each vertex has three edges coming from it and each edge has two vertices. So three times the number of vertices is twice the number of edges. So there are 120 edges.

Note that Euler's formula still holds since $80 - 120 + 42 = 2$. Alternatively this formula could have been used to determine the number of edges.

S4. Consider a rectangular grid of points with r rows and c columns. The distances between rows and between columns are exactly the same, and can be taken as one unit.

(a) Show that the area of the complete grid of points can be determined from $\frac{1}{2}E + I - 1$ where E is the number of points on the sides of the grid and I is the number of points within the grid.

(b) Now connect two opposite corner points of the rectangle, and consider one of the two right angled triangular regions. Show that exactly the same result holds for the area of one of these triangular regions, i.e. the area of the triangle is given by $\frac{1}{2}E + I - 1$ where E is the number of points on the sides of the triangle and I is the number of points within the triangle.

(c) Finally consider the triangular region formed when the two end points on one side of the rectangular grid are connected to one of the interior points of the opposite side. Again show that the area of the triangle is given by $\frac{1}{2}E + I - 1$ where E is the number of points on the sides of the triangle and I is the number of points within the triangle.

Solution

(a) The area of the rectangle is $(r - 1)(c - 1)$.

The total number of points on the edges of the rectangle is: $E = 2c + 2(r - 2)$ and the number of points within the rectangle is $I = (r - 2)(c - 2)$.

Hence $\frac{1}{2}E + I - 1 = c + (r - 2) + (r - 2)(c - 2) - 1 = (r - 1)(c - 1)$ which is the area of the rectangular region.

(b) Let the areas of the two triangles be T_1 and T_2 so $T_1 = T_2$. They also have the same number of points on their sides ($E_1 = E_2$) and the same number of internal points ($I_1 = I_2$). Further the sum of their areas, or twice the area of one triangle, is the area of the rectangular region, i.e. $T_1 + T_2 = 2T_1 = \frac{1}{2}E_R + I_R - 1$ where the subscript R indicates points for the complete rectangle.

Now let the number of points, excluding the end points, on the diagonal which is common to the two triangles, be d. Then equating numbers of points in the triangles and the rectangle, we get:

(interior points): $I_R = I_1 + I_2 + d$;

(edge points): $E_R + 2d + 2 = E_1 + E_2$.

Hence $2T_1 = \frac{1}{2}E_R + I_R - 1 = \frac{1}{2}\left(E_1 + E_2\right) - d - 1 + I_1 + I_2 + d - 1 = E_1 + 2I_1 - 2$ and $T_1 = \frac{1}{2}E_1 + I_1 - 1$, as required.

(c) The rectangular grid is divided into 3 triangles, A (the required triangle), B and C. Triangles B and C, which border A, are right-angled, so their areas are given by $T_B = \frac{1}{2}E_B + I_B - 1$ and $T_C = \frac{1}{2}E_C + I_C - 1$, respectively. Equating the area of the rectangle with the sum of the areas of the triangles gives: $\frac{1}{2}E_R + I_R - 1 = T_A + T_B + T_C$.

Let d_B and d_C be the numbers of interior points (so not on the edges of the rectangle) which are on the sides of A common to B and C, respectively. Equating interior and edge points (as in part(b)) gives:

(interior points): $I_R = I_A + I_B + I_C + d_B + d_C$;

(edge points): $E_R + 2\left(d_B + d_C\right) + 4 = E_A + E_B + E_C$.

Hence $T_A + T_B + T_C = \frac{1}{2}E_R + I_R - 1$

$= \frac{1}{2}\left(E_A + E_B + E_C - 2\left(d_B + d_C\right) - 4\right) + I_A + I_B + I_C + d_B + d_C - 1$

$= \frac{1}{2}E_A + I_A - 1 + \frac{1}{2}E_B + I_B - 1 + \frac{1}{2}E_C + I_C - 1$

$= \frac{1}{2}E_A + I_A - 1 + T_B + T_C$, so $T_A = \frac{1}{2}E_A + I_A - 1$ as required.

S5. In a Fantasy Football League, at a certain stage in the season, Hearts had lost some games but had won 8 games and drawn 9. Having played exactly the same number of games, Aberdeen had an average of 1.07 points per game. Which team was ahead at this stage? In this year, each team was playing a total of 36 league games in a season, there are 3 points for a win and 1 for a draw and the average number of points scored by Aberdeen is given correct to two decimal places.

Solution

Hearts had scored 33 points.

Now 33/1.07 is approximately 30.8.

Thus if the number of games played was less than 31, the average number of points per game achieved by Hearts would be greater than $33/30 = 1.1 > 1.07$, and so they would be ahead. Thus if Aberdeen were to be ahead, they must have played at least 31 games and the number of points scored divided by the number of games played must round off to 1.07.

Checking the fractions gives $33/31 = 1.06$, $34/31 = 1.10$. Thus $(n + 2)/n \geqslant 1.06$ for $n > 31$. Also check $39/36 = 1.08$ so that $(n + 3)/n \geqslant 1.08$ for $n < 36$. Thus no fractions with denominator between 36 and 31 rounds off to 1.07 and so Hearts must be ahead.