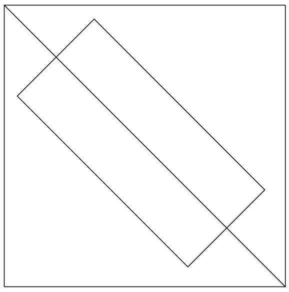
2008 Senior Set 1 solutions

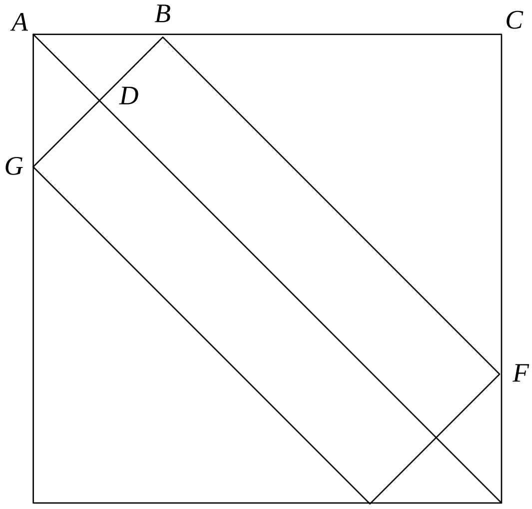
S1. Show that if a rectangle, which is twice as long as it is broad, can fit diagonally into a square as shown below, then it can also fit into the square with its sides parallel to the sides of the square.

Is this true if the rectangle is three times as long as it is broad? **Explain your answer.**



Solution

Consider the extreme case where the corners of the rectangle lie on the sides of the square.



Let the square have side length L. Let $BF = 2y$ and $BG = y$. So $BD = AD = \frac{1}{2}y$. So $AB^2 = AD^2 + BD^2 = \frac{1}{2}y^2$. So $AB = y/\sqrt{2}$.

Now $BC = CF$. So $2BC^2 = BF^2 = 4y^2$. So $BC = \sqrt{2}y$. So

$$L = AB + BC = \frac{y}{\sqrt{2}} + \sqrt{2}y = \frac{3}{\sqrt{2}}y.$$

So $BF = 2y = \frac{2\sqrt{2}}{3}L < L$ since $2\sqrt{2} < 3$.

So this extreme rectangle, and so any other rectangle which is twice as long as it is broad and which can fit in diagonally, could fit in with its sides parallel to the sides of the square.

Now suppose that $BF = 3y$. As before $AB = y/\sqrt{2}$ and now $2BC^2 = 9y^2$ so $BC = \frac{3}{\sqrt{2}}y$. So $L = \frac{4}{\sqrt{2}}y$. So $BF = 3y = \frac{3\sqrt{2}}{4}L$. Now $\frac{3\sqrt{2}}{4} > 1$ so this rectangle could *not* fit in with its sides parallel to the sides of the square.

S2. A queue of slow moving traffic is 5 miles long. It takes 15 minutes to pass a particular road sign. A police car takes a total of 20 minutes driving at constant speed to travel from the back of the queue to the front and to return to the back. How fast does the police car travel?

Solution

Let the speed of the police car be x mph.

It overtakes at $(x - 20)$ mph and returns at $(x + 20)$ mph.

$$\frac{5}{x - 20} + \frac{5}{x + 20} = \frac{1}{3}$$

$$\frac{5x + 100 + 5x - 100}{(x - 20)(x + 20)} = \frac{1}{3}$$

$$30x = x^2 - 400 \quad \Rightarrow \quad x^2 - 30x - 400 = 0$$

$$(x + 10)(x - 40) = 0 \quad \Rightarrow \quad x = -10 \text{ or } 40$$

But $x > 0$, hence the speed is 40mph.

S3. In 1946, an American numerologist, Professor Humbug, predicted the downfall of the USA in the year 2141 based on what he called a profound mathematical discovery depending on the following expression:

$$1492^n - 1770^n - 1863^n + 2141^n$$

He spent many months calculating the value of this for $n = 1, 2, 3$ and so on up to 1945 and found the remarkable fact that the result was always divisible by 1946. Since the years 1492, 1770 and 1863 are all important in American history, he claimed that 2141 would also be significant – hence his prediction.

Show how he could have saved himself months of work.

Hint: $(x^n - y^n) = (x - y)(x^{n-1} + \ldots + y^{n-1})$

Solution

$1492^n - 1770^n - 1863^n + 2141^n = (2141^n - 1863^n) - (1770^n - 1492^n)$
$= (2141^n - 1770^n) - (1863^n - 1492^n)$. Now
$(x^n - y^n) = (x - y)(x^{n-1} + x^{n-2}y + x^{n-3}y^2 + \ldots + y^{n-1})$. Thus is always divisible by $(x - y)$ for all values of n. So $2141^n - 1863^n$ is divisible by $2141 - 1863 = 278$.

$1770^n - 1492^n$ is divisible by $1770 - 1492 = 278$.

$2141^n - 1770^n$ is divisible by $2141 - 1770 = 371$.

$1863^n - 1492^n$ is divisible by $1863 - 1492 = 371$.

So the expression is always divisible by 278 and 371. Now $371 = 7 \times 53$ and $278 = 2 \times 139$. Since 7 does not divide 278 the expression will always be divisible by $7 \times 278 = 1946$.

S4. A circle of radius 15 cm intersects another circle of radius 20 cm at right angles. What is the difference of the areas of the non-overlapping portions? What is the sum of the areas of the non-overlapping portions?

Solution

On the diagram below, let X be the area of the overlapping portion. Then the difference in areas is

$$\left(20^2\pi - X\right) - \left(15^2\pi - X\right) = \pi\left(20^2 - 15^2\right) = 175\pi.$$

The sum of the areas is

$$\left(20^2\pi - X\right) + \left(15^2\pi - X\right) = \pi\left(20^2 + 15^2\right) - 2X.$$

Now X is equal to twice the area of the curved region RST on the diagram.

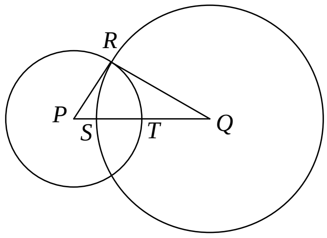
Area of Triangle PQR = area of circular sector PRT + area of circular sector QSR – area of curved region RST.

$\tan\angle RPQ = \dfrac{20}{15}$ and $\tan\angle RQP = \dfrac{15}{20}$. So

Area of curved region $RST = 20^2\pi\left(\dfrac{\tan^{-1}\frac{15}{20}}{360}\right) + 15^2\pi\left(\dfrac{\tan^{-1}\frac{20}{15}}{360}\right) - \dfrac{1}{2} \times 20 \times 15 = 83.021.$

So the sum of the areas of the non-overlapping portions is

$\pi\left(20^2 + 15^2\right) - 4 \times 83.021 = 1631.411.$



S5. Show that the square of any integer leaves a remainder of 0, 1, 4 or 7 when divided by 9. Use this to establish the following condition that a number which is a perfect square must satisfy the following:

> For a number that is a perfect square, add up its digits to form a second number. If that number has more than one digit, add up its digits to form a third number. Continue until you obtain a single digit number. That final number must be 1,4, 7 or 9.

Solution

We can write any integer in the form $9a + b$ where $0 \leqslant b < 9$. If we square this we get $(9a + b)^2 = 81a^2 + 18ab + b^2$. For the numbers 0 to 9, the remainder on dividing their squares by 9 are then 0,1,4,0,7,7,0,4,1. So the remainder on dividing the square of any number by 9 is 0,1, 4 or 7.

Suppose that $x^2 = a_r a_{r-1} \ldots a_1 a_0 = a_r 10^r + a_{r-1}10^{r-1} + \ldots + a_1 10 + a_0$
$= \left(a_r + a_{r-1} + \ldots + a_1 + a_0\right) + a_r\left(10^r - 1\right) + a_{r-1}\left(10^{r-1} - 1\right) + \ldots + a_1(10 - 1).$

Now for each t, $10^t - 1$ is divisible by 9. So $x^2 - \left(a_r + a_{r-1} + \ldots + a_1 + a_0\right)$ is divisible by 9.

Let $a_r + a_{r-1} + \ldots + a_1 + a_0 = b_s b_{s-1} \ldots b_1 b_0 = b_s 10^s + b_{s-1}10^{s-1} + \ldots + b_1 10 + b_0$
$= \left(b_s + b_{s-1} + \ldots + b_1 + b_0\right) + b_s\left(10^{s-1} - 1\right) + b_{s-1}\left(10^{s-1} - 1\right) + \ldots + b_1(10 - 1).$

So $x^2 - \left(b_s + b_{s-1} + \ldots + b_1 + b_0\right)$ is divisible by 9.

Now repeat until we get a number, call it f_0, with one digit. The remainder on dividing x^2 by 9 will be the same as the remainder on dividing f_0 by 9. So, by the first part, this number will be 1,4, 7 or 9. (Note that it cannot be 0.)