2007 Senior Set 2 solutions

S1. A cyclist and a runner start off simultaneously around a racetrack each going at a constant speed in the same direction. The cyclist completes one lap and then catches up with the runner. Instantly the cyclist turns around and heads back at the same speed to the starting point where he meets the runner who has just finished his first lap. Find the ratio of their speeds.

Solution

Let the speeds of the cyclist and the runner be u and v respectively. Let the fraction of the circuit covered by the runner when he meets the cyclist for the first time be x. At this time the cyclist will have covered $1 + x$ circuits. Equating the times gives:

$$\frac{x}{v} = \frac{1 + x}{u} \qquad (1)$$

$$\text{thus } ux = v(1 + x) \Rightarrow \frac{u}{v} = \frac{1 + x}{x}.$$

Similarly the time taken for the second phase:

$$\frac{1 - x}{v} = \frac{x}{u} \qquad (2)$$

$$\text{giving } u(1 - x) = vx \Rightarrow \frac{u}{v} = \frac{x}{1 - x}.$$

From the two equations, the ratio of u to v gives

$$\frac{1 + x}{x} = \frac{x}{1 - x}$$

Hence

$$x^2 = 1 - x^2.$$

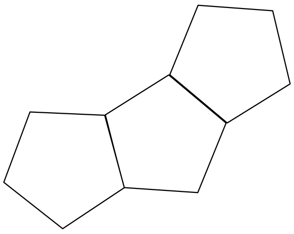
From this

$$x = \sqrt{\frac{1}{2}}$$

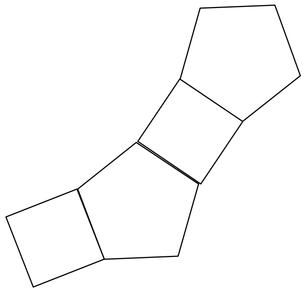
and this gives the ratio of the speeds as

$$\frac{u}{v} = \frac{1 + 1/\sqrt{2}}{1/\sqrt{2}} = \sqrt{2} + 1.$$

S2. Mahti has cut some regular pentagons out of card and is joining them together in a ring. How many pentagons will there be when the ring is complete?



She then decides to join the pentagons with squares which have the same edge length and wants to make a ring as before. Is it possible? If so, determine how many pentagons and squares make up the ring and if not, explain why.



Solution

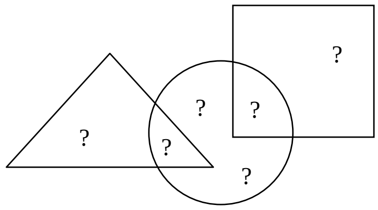
If $a_n°$ is the internal angle of a regular n-gon, then splitting it into n identical isosceles triangles we see that $na_n + 360 = 180n$ so that $a_n = 180\left(1 - \frac{2}{n}\right)$. So the internal angle of a regular pentagon is 108°. So the internal angle of the ring made out of pentagons is $(360 - 2 \times 108)°$ which is 144° which is the internal angle of a regular 10 gon. So there will be 10 pentagons in the ring.

For the pentagons and the squares, they will fit together in a ring if we can find a regular n-gon for some even number n whose internal angle is $b°$ where

$$b = 360 - (108 + 90) = 162 = 180\left(1 - \frac{2}{n}\right).$$

So $n = 20$. Thus they do form a ring which contains 10 pentagons and 10 squares.

S3. In the diagram, each question mark represents one of six consecutive whole numbers. The sum of the numbers in the triangle is 39, the sum of those in the square is 46 and the sum of those in the circle is 85. What are the six numbers?



Solution

Let the consecutive numbers be $a, a+1, a+2, a+3, a+4, a+5$.

Method 1.

The sum of the two numbers in the square less the sum of the two numbers in the triangle is 7. There are two ways this could arise.

Either $a + 4$ and $a + 5$ are in the square and a and $a + 2$ are in the triangle

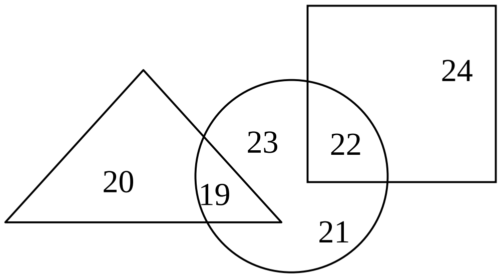
or $a + 3$ and $a + 5$ are in the square and a and $a + 1$ are in the triangle.

But the sum of the two numbers in the square is even so they must be $a + 3$ and $a + 5$.

So $2a + 8 = 46$, giving $a = 19$. So the four numbers in the circle are $a + 2$, $a + 4$ and one of $a + 3$, $a + 5$ and one of a, $a + 1$ i.e. they are 21, 23 and one of 22, 24 and one of 19, 20. Since their sum is 85 they must be 21, 23, 22, 19. Thus the numbers are distributed as shown below.

Method 2.

Let the number in the intersection of the circle and triangle be $a + t$ and the number in the intersection of the circle and the square be $a + s$. So the total of the six numbers is $6a + 15$ and that is equal to $39 + 46 + 85 - (a + t) - (a + s)$. So $6a + 15 = 170 - (2a + s + t)$. So $8a = 155 - (s + t)$. Now $s + t$ is between 1 and 9 and so $8a$ is between 154 and 146. Since a is a whole number this means that $a = 19$. Furthermore with $a = 19$ we get $s + t = 3$. Now the sum of the two smallest numbers is 39 so they must be in the triangle. The sum of the numbers in the square is 46 so they must be 22 and 24. So we must have $s = 3$ and $t = 0$. This gives which numbers lie in the intersections and hence all numbers as shown below.



S4. The triangle ABC is inscribed in a circle of radius 1. Show that the length of the side AB is given by $2\ \sin c°$, where $c°$ is the size of the interior angle of the triangle at C.

Solution

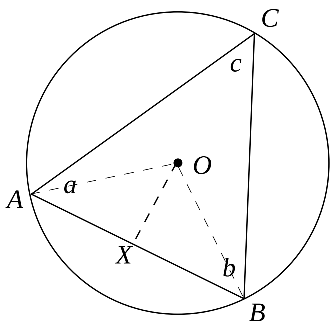


Fig. 1

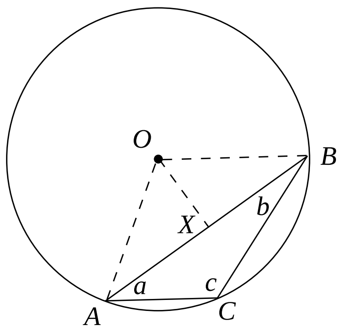


Fig. 2

1. Let the centre of the circle be O, and let the interior angles at the vertices A, B and C be a, b and c, respectively. (Clearly, from the sine rule, $AB = k \sin c$. It's a matter of determining the value of k).
2. Consider two situations: $\angle C$ is acute (Fig 1), and $\angle C$ is obtuse (Fig 2).
3. For both Figs: draw OA, OB (each length 1), and draw the perpendicular OX. Note that, since ΔAOB is isosceles, $AX = XB$ and $\angle AOX = \angle BOX$.
4. Fig 1: $\angle AOB = 2c$ (angle at centre is twice that at circumference from common chord – proof given below). Similarly, in Fig 2: $\angle AOB = 2(180 - c)$.
5. Fig 1: from triangle AOB, $AB = 2\ \sin\left(\frac{1}{2}\angle AOB\right) = 2\ \sin c$. Similarly, for Fig 2: from triangle AOB, $AB = 2\ \sin\left(\frac{1}{2}\angle AOB\right) = 2\ \sin(180 - c) = 2\ \sin c$.

Conclusion

The length of AB is $2\ \sin c$ as required.

Proof of 4: Consider a chord PQ of a circle centre O, and any diameter RS which cuts the chord inside the circle, where R lies on the shorter arc between P and Q. Angles SPR and SQR are both right angles. Let $x = \angle PRS$, $y = \angle QRS$, $u = \angle POS$ and $v = \angle QOS$. Triangles POR and QOR are isosceles, so $u = 2x$ and $v = 2y$. The angle at the centre is $\angle POQ = u + v$, and the angle at R, on the circumference, is $\angle PRQ = x + y$. Thus, $\angle POQ = u + v = 2x + 2y = 2(x + y) = 2\angle PRQ$.

S5. The number 3025 has the peculiar property that, if you split it into two parts as 30 and 25 then $(30 + 25)^2 = 3025$. Find all 4-digit numbers with this property.

Solution

We require two numbers c, d where $0 < c < 100$, $0 \leqslant d < 100$ such that $(c+d)^2 = 100c + d = (100-1)c + (c+d)$. So $(c+d)\big((c+d)-1\big) = 99c$.

Now $c + d$ and $(c + d) - 1$ are coprime. So factorise 99 into a product of two coprime integers a and b so that $99 = ab$. Thus $c + d = ax$, $(c + d) - 1 = by$ for some $x > 0$ and $y > 0$. Note then that $c = xy$ and $d = x(a - y)$.

Now $ax = by + 1$. Solve this equation with $0 < x < b$, $0 < y < a$ for each pair of possible coprime factors of 99.

(a) $a = 11$, $b = 9$. So $x = 5$, $y = 6$, $c = 30$, $d = 25$. This gives the given example 3025.

(b) $a = 9$, $b = 11$. So $x = 5$, $y = 4$, $c = 20$, $d = 25$. This gives the solution 2025.

(c) $a = 99$, $b = 1$. So $x = 1$, $y = 98$, $c = 98$, $d = 1$. This gives the solution 9801.

(d) $a = 1$, $b = 99$. Then $ax = by + 1$ has no solution with $0 < x < 99$, $0 < y < 1$.

Thus there are precisely three numbers: 3025; 2025 and 9801, with the required property.

Comment

This is a beautiful solution but this "out-of-the-box" approach is quite hard. This problem can also be done by using

$10^3a + 10^2b + 10c + d = 10^2(a + c)^2 + 20(a + c)(b + d) + (b + d)^2$

for $0 \leqslant a, b, c, d < 10$. First show that there are only 10 possible pairs for (b, d) and also that $2 \leqslant a + c \leqslant 9$. Then a very tedious enumeration completes the solution.