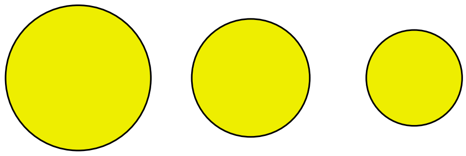
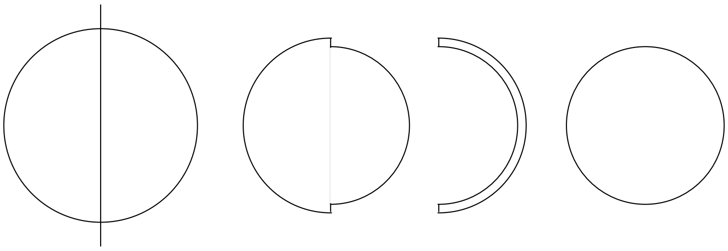
2007 Senior Set 1 solutions

S1. Three large pancakes, as shown, each of the same thickness are to be shared equally among four people. The diameters of the pancakes form a Pythagorean triple. Without measuring, show how to cut the pancakes to make a total of just five pieces so that each person can get the same amount.
Explain your reasoning.



Solution

If the diameters are a, b, c then $a^2 = b^2 + c^2$. Since the pancakes are all of the same thickness, the amount of pancake is proportional to its area. The areas of the pancakes are $\frac{1}{4}\pi a^2$, $\frac{1}{4}\pi b^2$, $\frac{1}{4}\pi c^2$ so the total area is $\frac{1}{2}\pi a^2$. So cutting the largest pancake in half gives two of the four equal portions. Now place the smallest pancake over the middle one and cut off half of the portion between the two pancakes. That has area $\frac{1}{2}\left(\frac{1}{4}\pi b^2 - \frac{1}{4}\pi c^2\right)$. That part together with smallest circle then has total area $\frac{1}{2}\left(\frac{1}{4}\pi b^2 + \frac{1}{4}\pi c^2\right) = \frac{1}{8}\pi a^2$. Also the part that remains on the middle circle has the same area so we get four equal portions.



S2. In Tiffany's, a world famous jewellery store, there is a string necklace of 33 pearls. The middle one is the largest and most valuable. The pearls are arranged so that starting from one end, each pearl is worth \$100 more than the preceding one, up to the middle one; and starting from the other end, each pearl is worth \$150 more than the preceding one, up to the middle one. If the total value of the necklace is \$65,000 what is the value of the largest pearl?

Solution

Let \$$m$ be the value of the first pearl at one end and \$$n$ the value of the pearl at the other end. So, counting from one end, the value of the middle pearl is $m + 16 \times 100$. Counting from the other end, the value of the middle pearl is $n + 16 \times 150$.
Thus $m + 16 \times 100 = n + 16 \times 150$ so $m = n + 800$.
The total value of the necklace is the value of the middle pearl plus the other pearls whose value is

$$m + (m + 100) + (m + 2 \times 100) + \ldots + (m + 15 \times 100)$$

$$+ n + (n + 150) + (n + 2 \times 150) + \ldots + (n + 15 \times 150)$$

$$= 16(m + n) + (100 + 150)(1 + 2 + 3 + \ldots + 15) = 16(m + n) + 30000.$$

So the total value of the necklace is $16(m+n) + 30000 + m + 1600$. Eliminating m we get

$$16(2n + 800) + 30000 + n + 800 + 1600 = 65000.$$

Solving this equation for n we get $33n + 45200 = 65000$ which gives $n = 600$.
Thus the value of the largest pearl is $600 + 16 \times 150 = 3000$.

S3. Find the ten-digit number which uses each of the digits 0 to 9 such that the numbers formed

by the first digits is divisible by 1,
by the first two digits is divisible by 2,
by the first three digits is divisible by 3,
by the first four digits is divisible by 4,
by the first five digits is divisible by 5,
by the first six digits is divisible by 6,
by the first seven digits is divisible by 7,
by the first eight digits is divisible by 8,
by the first nine digits is divisible by 9,
by the first ten digits is divisible by 10.

Solution

Denote the number by $abcdefghij$. Since it is divisible by 10, clearly $j = 0$.

Also since $abcde$ is divisible by 5, then $e = 5$. Since ab, $abcd$, $abcdef$, $abcdefgh$ are all divisible by 2, then b, d, f, h must all be even numbers. Thus in some order, b, d, f, h are the digits 2,4,6,8 and, also in some order, a, c, g, i are the digits 1, 3, 7, 9.

Recall that a number is divisible by 3 if and only if the sum of its digits is divisible by 3. So, since $abcdefghi$ is divisible by 3 and $abcdef$ is also divisible by 3, it follows that $g + h + i$ is divisible by 3. In the same way, $d + e + f$ will be divisible by 3 and also $a + b + c$ will be divisible by 3.

Now $abcd = ab \times 100 + cd$ is divisible by 4 and since 100 is divisible by 4, then cd must be divisible by 4. In a similar way, since f is even, $abcdefgh$ = (an even number) $\times 100 + gh$. But 200 is divisible by 8, it follows that gh is divisible by 8.

To summarize this, we need to find b, d, f, h among the numbers 2, 4, 6, 8 and a, c, g, i among the numbers 1, 3, 7, 9 such that the following conditions hold.

1. gh is divisible by 8.
2. $g + h + i$ is divisible by 3.
3. cd is divisible by 4.
4. $d + 5 + f$ is divisible by 3.
5. $abcdefg$ is divisible by 7.

First find the possibilities which satisfy 1. and 2. These are :

g	3	7	7	9	3
h	2	2	2	6	2
i	7	3	9	3	1
	A	*B*	*C*	*D*	*E*

For each of these find solutions to 3. and 4. and finally check 5.

A.	c	1	9	
	d	6	6	These yield 9816543270 and 1896543270, both of which fail 5.
	f	4	4	

B.	c	1	9	
	d	6	6	These yield 9816547230 and 1896547230, both of which fail 5.
	f	4	4	

C.	c	1	3	
	d	6	6	These yield 3816547290 which does satisfy 5. and 1836547290 which does not.
	f	4	4	

D.	c	1	7
	d	2	2
	f	8	8

E.	c	7	9	
	d	6	6	These yield 9876543210 and 7896543210 both of which fail 5.
	f	4	4	

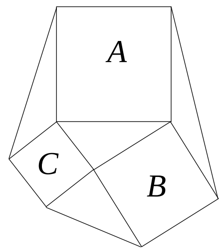
It is now easy to check that 3816547290 satisfies all the conditions.

S4. A farmer owns three square fields of areas A, B, C which are located as shown in the diagram.

He then buys the four triangular plots of land shown so that he can put a fence with six straight edges round his property.

Show that the area of each of the triangular plots of land is the same.

Further, if A is 26 acres, B is 20 acres and C is 18 acres, find the total area included inside the farmer's fence.



Solution

Let the side of square A have length a, B have length b and C have length c.

Area of any triangle is $\frac{1}{2}$ base × altitude.

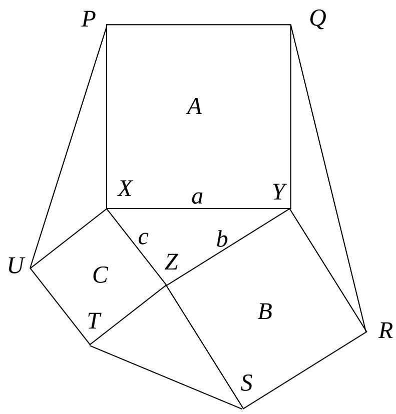
Area of triangle $XYZ = \frac{1}{2}ab\sin\angle Y = \frac{1}{2}bc\sin\angle Z$
$= \frac{1}{2}ca\sin\angle X$.

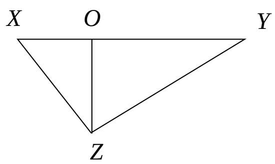
Area of triangle $QYR = \frac{1}{2}ab\sin(180° - \angle Y)$
$= \frac{1}{2}ab\sin\angle XYZ$.

Area of triangle $SZT = \frac{1}{2}bc\sin(180° - \angle Z)$
$= \frac{1}{2}bc\sin\angle YZX$.

Area of triangle $UXP = \frac{1}{2}ca\sin(180° - \angle X)$
$= \frac{1}{2}ca\sin\angle ZXY$.

So all four triangles have the same area.





Consider triangle XYZ.

$XY = a$, $YZ = b$, $ZX = c$. Let $OZ = h$ and $OY = x$ so that $OX = a - x$. Then from the two right-angled triangles we have $h^2 + x^2 = b^2$ and $h^2 + (a - x)^2 = c^2$.

Subtracting gives $a^2 - 2ax = c^2 - b^2$. So $x = \dfrac{a^2 + b^2 - c^2}{2a}$

and $h^2 = b^2 - x^2 = \dfrac{4a^2b^2 - (a^2 + b^2 - c^2)^2}{4a^2}$.

Area of triangle $XYZ = \frac{1}{2}ah = \frac{1}{4}\sqrt{4a^2b^2 - (a^2 + b^2 - c^2)^2}$.

For the figures given this gives, in acres, $\frac{1}{4}\sqrt{4 \times 26 \times 20 - (26 + 20 - 18)^2} = 9$. So total area is 100 acres.

[They probably don't know about acres, but notice that you do not need to].

S5. You are given three positive whole numbers whose sum is M. If you subtract $\frac{1}{3}M$ from the first one, add 2 to the second one and multiply the third by 2, you get the same set of three numbers back again. Work out all possible values for the three numbers.

Solution

(There are two possible solutions to this).

Let the numbers be a, b, c. So $a + b + c = M$. The set $\{a, b, c\}$ is the same as the set $\{a - \frac{1}{3}M, b + 2, 2c\}$. So $a + b + c = a - \frac{1}{3}M + b + 2 + 2c$. This gives $c = \frac{1}{3}M - 2$. So either $a = 2c$ or $b = 2c$.

Case 1: $a = 2c$

So $a = \frac{2}{3}M - 4$ and $a + c = M - 6$. So $b = 6$ and $b + 2 = c = 8$. Thus $a = 16$. The sum of the three numbers is then 30 and $a - 10 = b$, $b + 2 = c$, $2c = a$.

Case 2: $b = 2c$

So as above $a = 6$ and $b + 2 = a$. Thus $b = 4$ and $c = 2$. The sum of the three numbers is 12 and $a - 4 = c$, $b + 2 = a$, $2c = b$.

So (a, b, c) is either (16,6,8) or (6,4,2).