2006 Senior Set 1 solutions

S1. In a snowball 'fight', where snowballs are identical spheres, your opponents have stacked their snowballs in a square pyramid. You are about to count the snowballs along the bottom edge of the opponent's stack when one appears with another snowball. After giving him a telling off, the opposition's leader takes apart the square pyramid and builds a new, triangular pyramid using all the original snowballs and the extra one. Find two possible values for the number of snowballs that your opponents now have.

Solution

The total in the square pyramid is $1^2 + 2^2 + 3^2 + \ldots + n^2 = S$.
By a known result $S = \frac{1}{6}n(n+1)(2n+1)$.

The total in the triangular pyramid is $\frac{1 \times 2}{2} + \frac{2 \times 3}{2} + \frac{3 \times 4}{2} + \ldots + \frac{m(m+1)}{2} = T$.

Hence

$$\begin{aligned} 2T &= 1 \times (1+1) + 2 \times (2+1) + 3 \times (3+1) + \ldots + m(m+1) \\ &= \left(1^2 + 2^2 + 3^2 + \ldots + m^2\right) + \left(1 + 2 + 3 + \ldots + m\right) \\ &= \frac{1}{6}m(m+1)(2m+1) + \frac{1}{2}m(m+1) \\ &= \frac{1}{6}m(m+1)\left[2m+1+3\right] = \frac{1}{3}m(m+1)(m+2) \end{aligned}$$

i.e. $$T = \frac{1}{6}m(m+1)(m+2)$$

So for some values of n, m we require that $S + 1 = T$.
This holds for $n = 5$, $m = 6$ in which case the number of snowballs is 56. It also holds for $n = 9$, $m = 11$ in which case the number of snowballs is 286.
[In general it needs a solution to $n(n+1)(2n+1) + 6 = m(m+1)(m+2)$.
That may well have infinitely many solutions.]

Tabulating might be more accessible:

	1	2	3	4	5	6	7	8	9	10	11	
S	1	5	14	30	55	91	140	204	285	…		
T	1	4	10	20	35	56	84	120	165	220	286	…

S2. Three cyclists are out for the day. Two are on a tandem and one on an ordinary cycle. Disaster struck when the ordinary cycle was stolen while they were having lunch in a café. They were left with the tandem and 20 miles to go. The tandem has to have two riders and the third person walks. Anne can walk a mile in 20 minutes, Sam in 30 minutes and Oscar in 40 minutes. The tandem travels at 20 miles per hour no matter which pair is riding it. What is the shortest time for all three to get home?

Solution

The method is that the tandem and a walker start off together. The tandem goes a certain distance, stops and one gets off and starts to walk. The other stays with the tandem to wait for the walker to catch up. Then the two get on and cycle after the walker who was originally on the tandem. When they catch up with this walker, the whole method starts again. Since Oscar is the slowest walker, he should always be on the bike. So either Anne or Sam start walking. For shortest time, the change over should be made so that the tandem and the second walker arrive home together.

So let tandem cycle for n minutes and so will cover $n/3$ miles. If Anne starts walking first, her total time to get home will be $\frac{\frac{n}{3}}{\frac{1}{20}} + \frac{\left(20 - \frac{n}{3}\right)}{\frac{1}{3}} = 60 + \frac{17n}{3}$. The total time for Sam will be $n + \frac{\left(20 - \frac{n}{3}\right)}{\frac{1}{30}} = 600 - 9n$. So we require that $600 - 9n = 60 + \frac{17n}{3}$. This gives that $n = \frac{3 \times 540}{44}$. So the total journey time works out to be 268.6 minutes i.e. 4 hours and 28.6 minutes.

S3. A group of seven girls − Ally, Bev, Chi-chi, Des, Evie, Fi and Grunt – were playing a game in which the counters were beans. Whenever a girl lost a game, from her pile of beans she had to give each of the other girls as many beans as they already had. They had been playing for some time and they all had different numbers of beans. They then had a run of seven games in which each girl lost a game in turn, in the order given above. At the end of this sequence of games, amazingly, they all had the same number of beans – 128. How many did each of them have at the start of this sequence of seven games?

Solution

Notice that the total number of beans they had did not change with each game. So that total was $T = 7 \times 128$.

At the end of each game, if the girl who lost had n beans at the start of the game, then she had $2n - T$ beans at the end of that game and all the others had twice as many as when they started that game.

Thus, suppose that the number of beans each girl, in the order given above, had at the start of the run of seven games was A, B, C, D, E, F, G. After the next game they had, in order, $2A - T$, $2B$, $2C$, $2D$, $2E$, $2F$, $2G$. After the next game they had, in order, $2^2A - 2T$, $2^2B - T$, 2^2C, 2^2D, 2^2E, 2^2F, 2^2G. After the next game they had, in order, $2^3A - 2^2T$, $2^3B - 2T$, $2^3C - T$, 2^3D, 2^3E, 2^3F, 2^3G. Continuing in this way after seven games they had, in order, $2^7A - 2^6T$, $2^7B - 2^5T$, $2^7C - 2^4T$, $2^7D - 2^3T$, $2^7E - 2^2T$, $2^7F - 2T$, 2^7G. Now $T - 7 \times 128$ and each of these numbers is $128 = 2^7$.

So $2^7G - 7 \times 2^7 = 2^7$. So $G = 8$. In the same way $2^7 - 2 \times 7 \times 2^7 = 2^7$ so that $F = 15$. All the others can be deduced in the same way so that we obtain, $E = 29$, $D = 57$, $C = 113$, $B = 225$, $A = 449$.

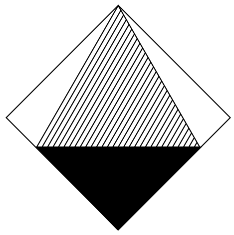
Alternative

Work backwards in an array:

128	128	128	128	128	128	128
64	64	64	64	64	64	512
32	32	32	32	32	480	256

etc.

S4. Pat and Jo were having a holiday in Greece and visited a temple. They noticed a wall tile as shown alongside.

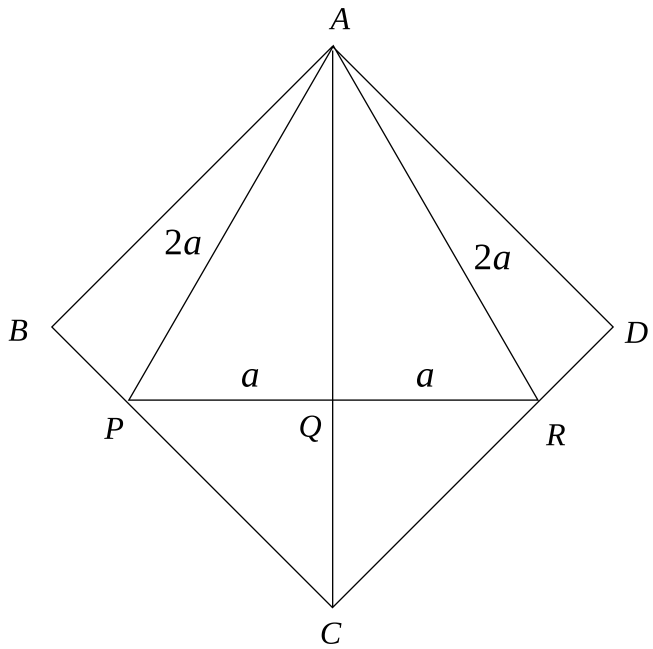


The shaded triangle is equilateral with one of its vertices at one corner of the surrounding square and the others on two of the sides of the square. Pat said "I bet the area of the black triangle is the same as the combined areas of the two white triangles". "Don't be daft!" said Jo.

Who was right and why?

Solution

(One of several possibilities.)



Let the sides of the equilateral triangle be of length $2a$. Using Pythagoras' Theorem, $AQ = \sqrt{4a^2 - a^2} = a\sqrt{3}$ so $\Delta APR = \frac{1}{2}(2a)(a\sqrt{3}) = a^2\sqrt{3}$.

Now consider the isosceles triangle CPR. Since $\angle PCR = 90°$, the circle with PR as diameter passes through C so $CQ = a$ and $\Delta CPR = \frac{1}{2}(2a)a = a^2$.

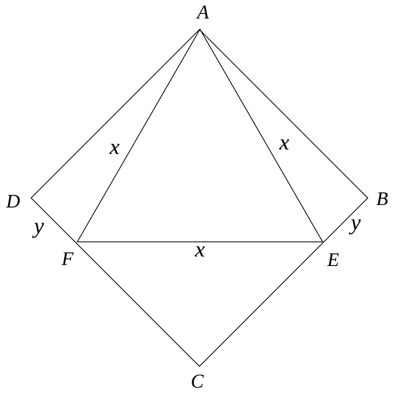
We now calculate the area of the square $ABCD$ by treating it as a rhombus. Recall that the area of a rhombus is half of the product of its diagonals. Thus

$$\text{area } ABCD = \tfrac{1}{2}AC.BD = \tfrac{1}{2}AC^2$$

$$= \tfrac{1}{2}\left(a + a\sqrt{3}\right)^2 = \tfrac{1}{2}a^2\left(4 + 2\sqrt{3}\right) = \left(2 + \sqrt{3}\right)a^2$$

So the combined area of the two white triangles = $a^2\left(2 + \sqrt{3} - 1 - \sqrt{3}\right) = a^2$ which is the same as the area of ΔCPR. So Pat was right!

A very neat alternative



Sum of areas of white triangles

$= 2 \times \frac{1}{2} \times yz = yz$ (1)

Area of black triangle = $\frac{1}{2}(z - y)^2$ (2)

But, by Pythagoras' theorem

$$2(z - y)^2 = x^2 = y^2 + z^2$$

$$\text{so} \quad y^2 + z^2 = 4yz$$

So (1) is yz and (2) is

$\frac{1}{4} \times 2(z - y)^2 = \frac{1}{4}\left(y^2 + z^2\right) = yz.$

S5. Let a, b, c be the lengths of the sides of a triangle. Show that

$$\frac{a}{b + c} + \frac{b}{a + c} + \frac{c}{a + b} < 2.$$

Describe the shape of a triangle for which the expression, $\frac{a}{b+c} + \frac{b}{a+c} + \frac{c}{a+b}$, is very close to 2.

Solution

Since $a + b > c$, $2(a + b) > c + a + b$ so $\frac{1}{a + b} < \frac{2}{a + b + c}$.

Similarly for others. So

$$\frac{a}{b + c} + \frac{b}{a + c} + \frac{c}{a + b} < \frac{2a + 2b + 2c}{a + b + c} = 2.$$

Alternative

Suppose that $a \leqslant b \leqslant c$. Since these are the lengths of the sides of a triangle $a + b > c$. Now consider the first two terms of the left-hand side

$$\frac{a}{b + c} + \frac{b}{a + c} = \frac{a(a + c) + b(b + c)}{(b + c)(a + c)} = \frac{a^2 + b^2 + c(a + b)}{ab + c^2 + c(a + b)}.$$

Since $a \leqslant b$, $a^2 \leqslant ab$ and since $b \leqslant c$, $b^2 \leqslant c^2$. So $\frac{a^2 + b^2 + c(a + b)}{ab + c^2 + c(a + b)} \leqslant 1$. Also $\frac{c}{a + b} < 1$. So $\frac{a}{b + c} + \frac{b}{a + c} + \frac{c}{a + b} < 2$.

If we make the triangle so that it has one very short side, so that a is close to 0, and the other two sides long, then the expression is very close to 2. For then the expression $\frac{a}{b + c}$ is very small and b and c will be almost equal. Thus each of the expressions $\frac{b}{a + c}$ and $\frac{c}{a + b}$ will be very close to 1 and the whole expression very close to 2.