2006 Senior Set 2 solutions

S1. Daryl put some black stones and some white stones into a bag. He then asked Ran to reach into the bag, without looking in, and draw out a stone. Ran drew out a black stone. Daryl asked Ran to draw out another stone and, once again, Ran drew out a black stone.

"There must be more black than white stones in the bag ," said Ran. "I wonder what the probability is of my drawing a black stone on a third try?"

Daryl replied," Exactly nine tenths of what it was of drawing a black stone on your first draw."

Daryl told Ran that he had put "ten, give or take two or three" stones into the bag. How many stones were in the bag at the start?

Solution

Suppose that there are n stones in the bag, of which b are black and w are white.

The probability of drawing a black stone on the first draw is $\dfrac{b}{b+w}$ and after drawing 2 black balls it is $\dfrac{b-2}{b+w-2}$. Hence,

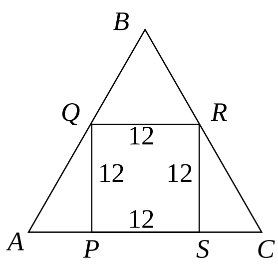
$$\frac{b-2}{b+w-2} = \frac{9}{10}\frac{b}{b+w}.$$

This gives $10b^2 + 10bw - 20b - 20w = 9b^2 + 9bw - 18b$,

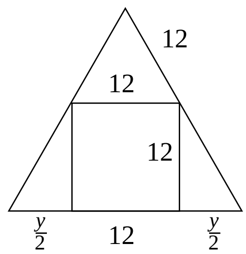
We know that $w = n - b$ and n is between 7 and 13; substituting and simplifying gives $b = \dfrac{20n}{n+18}$.

The only value of n that makes b an integer for n between 7 and 13 is, where $n = 12$, where $b = 8$. This makes $w = 4$.

S2. *PQRS* is a square of side 12 cm which fits inside an equilateral triangle *ABC* in such a way that each of P, Q, R and S is on a side of *ABC* as shown. Calculate the area of triangle *ABC*.



Solution



$\tan 30° = \frac{y/2}{12}$ so $y = 24\tan 30° = 8\sqrt{3}$.

Side of triangle = $y + 12 = 8\sqrt{3} + 12$.

Height of triangle = $12 + 6\sqrt{3}$.

Area of triangle

$= \frac{1}{2}\left(8\sqrt{3} + 12\right)\left(12 + 6\sqrt{3}\right)$

$= 12\left(2\sqrt{3} + 3\right)\left(2 + \sqrt{3}\right)$

$= 12\left(12 + 7\sqrt{3}\right)$

S3. Let $x_1, x_2, \dots, x_{100}$, denote 100 positive numbers whose sum is 100.

(i) Explain why there must be at least one suffix i such that $x_i \leqslant 1$.

(ii) Explain why there must be at least one suffix i such that $x_i + x_{i+1} \leqslant 2$.

(iii) Find 100 such positive numbers, all different, such that $x_i + x_{i+1} \leqslant 2$ for all i.

Solution

(i) Suppose that this is not true. So, for all i, we must have $x_i > 1$. But then the sum over all 100 will exceed 100. That is false so there must be at least one i such that $x_i \leqslant 1$.

(ii) Suppose this is not true. So, for all i we must have $x_i + x_{i+1} > 2$. In particular, for all i, $x_{2i-1} + x_{2i} > 2$. Now add up for $i = 1,2, \dots ,50$ to get that the sum of all x_i exceeds 100. Again a contradiction.

(iii) Let $x_{2i-1} = 1 + \dfrac{1}{1+i}$, $x_{2i} = 1 - \dfrac{1}{1+i}$. Clearly each $x_i > 0$ and $x_{2i-1} + x_{2i} = 2$. Also

$$x_{2i} + x_{2i+1} = 1 - \frac{1}{1+i} + \left(1 + \frac{1}{i+2}\right) = 2 - \frac{1}{(i+1)(1+2)} < 2.$$

By construction, all numbers are different.

S4. A dishonest barman removes three litres of wine from a barrel, replacing them with water and mixing the contents. He repeats the theft twice more, removing in total 9 litres and replacing them with water. As a result the wine remaining in the barrel is half its former strength. How much wine did the barrel originally hold? (Your answer may given to two decimal places.)

Solution

Let the barrel hold x litres.

Stage	Wine in barrel before	Wine removed	Wine in barrel after
1	x	3	$x - 3$
2	$x - 3$	$\frac{3}{x}(x-3)$	$\frac{(x-3)^2}{x}$
3	$\frac{(x-3)^2}{x}$	$\frac{3}{x}\frac{(x-3)^2}{x}$	$\frac{(x-3)^3}{x^2}$

Thus
$$\begin{aligned}
\frac{(x-3)^3}{x^2} &= \frac{x}{2}\\
\left(\frac{x-3}{x}\right)^3 &= \frac{1}{2}\\
\left(1 - \frac{3}{x}\right)^3 &= \frac{1}{2}\\
1 - \frac{3}{x} &= \frac{1}{\sqrt[3]{2}}\\
\frac{3}{x} &= 1 - \frac{1}{\sqrt[3]{2}}\\
\frac{1}{x} &= \frac{1}{3}\left(1 - \frac{1}{\sqrt[3]{2}}\right) \approx 0.06876\\
x &\approx 14.54
\end{aligned}$$

The barrel held about 14.54 litres of wine.

S5. Find the largest integer which is both an n-digit number and an nth power.

Solution

10^n has $n + 1$ digits, so if a^n has n digits then $a < 10$. Also if $a > b$ and b^n has n digits, then a^n has n digits. So we need to find the largest n such that 9^n has n digits. (For the largest n, both 9^n and 9^{n+1} will have n digits. This means we need to find the largest n such that $9^n \geqslant 10^{n-1}$ (or the smallest n such that $9^{n+1} < 10^n$).) One can do this by direct calculation using a sufficiently accurate calculator. Alternatively it can be done using logs if they are familiar with that. For we require to find n such that $9^n > 10^{n-1}$. i.e. $9 > (\frac{10}{9})^{n-1}$. So $\operatorname{Log} 9 > (n-1)\operatorname{Log}(10/9)$.

Thus $n - 1 < \operatorname{Log} 9 / \operatorname{Log}(10/9) = 20.8$. Thus $n \leqslant 21$. Thus the required number is 9^{21} which is 109,418,989,131,512,359,209.