Middle Division: Problems 2

M1. Imran bought a cat and dog for £60 each. Later he sold them. He made a profit of 20% on the dog. He made a loss of 20% on the cat.

How much did Imran get altogether when he sold the cat and dog?

Later Andy bought another cat and dog. He sold them for £60 each. He made a profit of 20% on the dog. He made a loss of 20% on the cat.

Did Andy make a profit or loss on the whole deal? If so, how much?

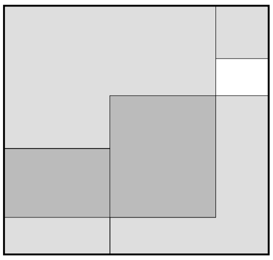
Solution

The 20% of £60 gain exactly balances the 20% of £60 loss, so Imran gets £120 exactly.

Andy sold his dog for £60, which was 120% = 6/5 of what he paid. So he paid 5/6 of £60 = £50.
Andy sold his cat for £60, which was 80% = 4/5 of what he paid. So he paid 5/4 of £60 = £75.

So Andy paid out £125 but only received £120, i.e. he made a loss of £5.

M2.



Four square sheets of tinted glass with sides 10 cm, 20 cm, 30 cm and 40 cm are placed in a rectangular box as shown. 70 square cm of the base of the box are left uncovered (white in the diagram).

What is the area of overlap of the sheets of glass (dark grey in the diagram)?

Solution

The smallest square is 10cm by 10cm so the longer side of the white area is 10 cm. So, as its area is $70\ \text{cm}^2$, its shorter side is $\frac{70}{10} = 7$ cm.

So the top edge in the diagram is $40\ \text{cm} + 10\ \text{cm}\ (= 50\ \text{cm})$ (or the same from the bottom edge).

The side on the right is $(10 + 7 + 30)\ \text{cm} = 47\ \text{cm}$.

So the area of the whole rectangle is $50\ \text{cm} \times 47\ \text{cm} = 2350\ \text{cm}^2$ of which $2350 - 70 = 2280\ \text{cm}^2$ is covered by at least one layer of glass.

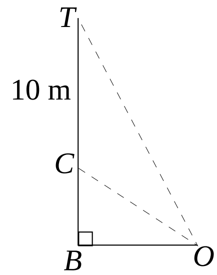
But the total area of the glass is

$$40 \times 40 + 30 \times 30 + 20 \times 20 + 10 \times 10 = 3000\ \text{cm}^2.$$

So $3000 - 2280 = 720\ \text{cm}^2$ must be in a double layer.

M3. An observer (O) is watching a climber scaling a vertical rock face from a point level with the base of the rock face (B) but some distance away. The observer looks up at an angle of 60° to see the top of the rock face. The observer looks up at an angle of 30° to see the climber (C) when the climber is 10 metres from the top.

How high is the rock face?



Solution

$$\angle CTO = \angle BTO = 90° - \angle BOT = 90° - 60° = 30°$$

$$\angle TOC = \angle TOB - \angle COB = 60° - 30° = 30°$$

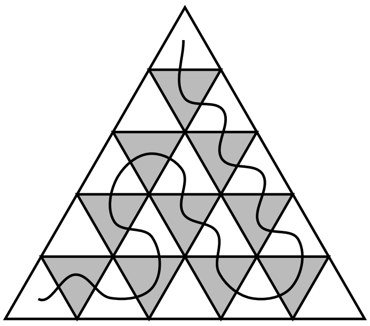
So triangle CTO is isosceles and so $CO = CT = 10$ m.

$$CB = CO \sin \angle COB = 10 \sin 30° = 10 \times 0.5 = 5 \text{ m}$$

So the height of the rock face is $BC + CT = 5 + 10 = 15$ m.

M4. A large equilateral triangle with sides of integer length N is split into small equilateral triangular cells each with side length 1 by drawing lines parallel to its sides. A continuous track starts in the cell at one corner of the large triangle and moves from cell to cell, always crossing at an edge shared by the two cells. The track never revisits a cell. Find, with proof, the greatest number of cells that can be visited on one track.

Solution



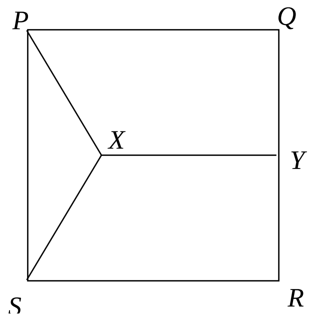
Colour the cells alternately grey and white as shown when $N = 5$. For a large triangle with side N there are always T_N white cells and T_{N-1} grey cells, where T_N is the N th triangular number. $T_N - T_{N-1} = N$ so there are N more white cells than grey cells.

Every possible track alternates between white and grey cells, always starting on a white cell. So once all the grey cells have been visited, with a track starting and ending on a white cell, there must be $N - 1$ white cells remaining. A track of this length is always possible, by missing the final white cell at the end of each row, apart from the last cell (i.e. missing $N - 1$ white cells in all), as shown in the diagram.

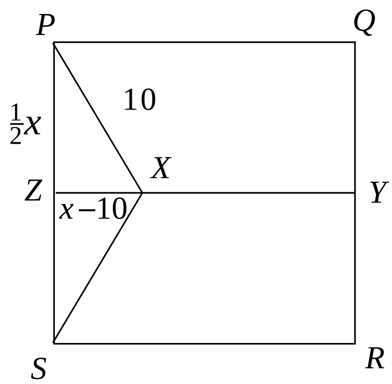
The longest track passes through T_{N-1} grey cells and $T_{N-1} + 1$ white cells, where $T_{N-1} = \frac{N}{2}(N - 1)$.

Hence the maximum track length is $N(N - 1) + 1$.

M5. In the diagram, $PQRS$ is a square with XY perpendicular to QR and $XP = XS = XY = 10$ cm. What is the area of the square?



Solution



Let the side of the square be x cm.
Construct XZ perpendicular to PS.

Apply Pythagoras' theorem to the right-angled triangle PXZ

$$ZX^2 + ZP^2 = PX^2$$

$$(x - 10)^2 + \left(\frac{x}{2}\right)^2 = 10^2$$

$$x^2 - 20x + 100 + \frac{x^2}{4} = 100$$

$$\frac{5x^2}{4} - 20x = 0$$

$$x(5x - 80) = 0$$

So either $x = 0$ (the triangle collapses, which it clearly does not) or $x = 16$.

So the area of the square is $16^2 = 256$ cm^2.