2020-2021 Middle Division: Problems 1: Solutions

M1. Jo is going on an 8-day activity holiday. Each day she can choose one of the water sports: kayaking or sailing, or land-based sports. She never does different water sports on consecutive days. She also wants to try all three options on at least one day of her holiday.

How many different schedules are possible?

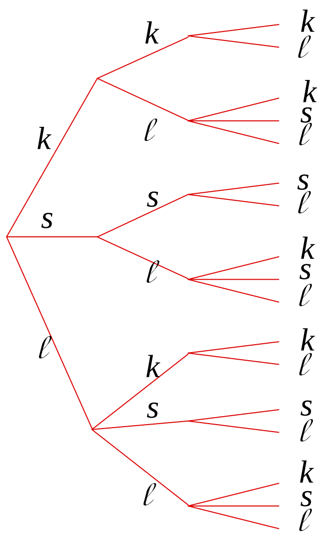
Solution

On the first day all 3 choices are possible.

After kayaking, only 2 choices are possible: kayaking or land-based. After sailing, only 2 choices are possible: sailing or land-based.

After land-based, all 3 choices are possible.

So we build up a tree diagram:



Day 1 choices: $k + s + \ell = 3$

Day 2 choices: $2k + 2s + 3\ell = 7$

Day 3 choices: $5k + 5s + 7\ell = 17$

Day 4 choices: $12k + 12s + 17\ell = 41$

Day 5 choices: $29k + 29s + 41\ell = 99$

Day 6 choices: $70k + 70s + 99\ell = 239$

Day 7 choices: $169k + 169s + 239\ell = 577$

Day 8 choices: $408k + 408s + 577\ell = 1393$

Note that this total includes holidays which have only one or two different activities.

It is not possible to have a schedule with only kayaking and sailing.

With only sailing and land-based there are $2^8 = 256$ possible schedules.

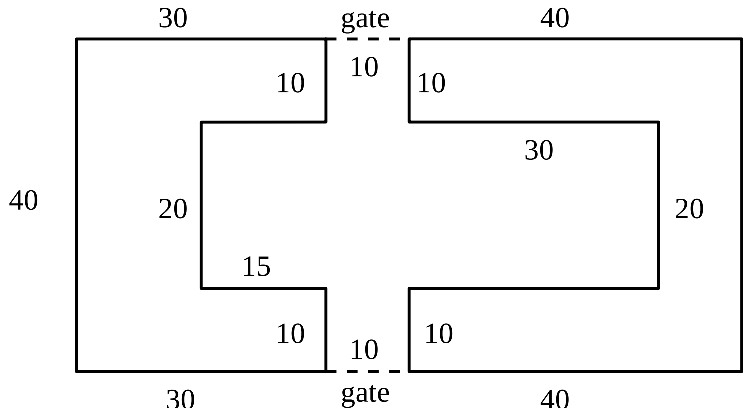
With only kayaking and land-based there are also $2^8 = 256$ possible schedules.

The schedule with only land-based is in both groups.

So there are $256 \times 2 - 1 = 511$ schedules with 2 or 1 different activities.

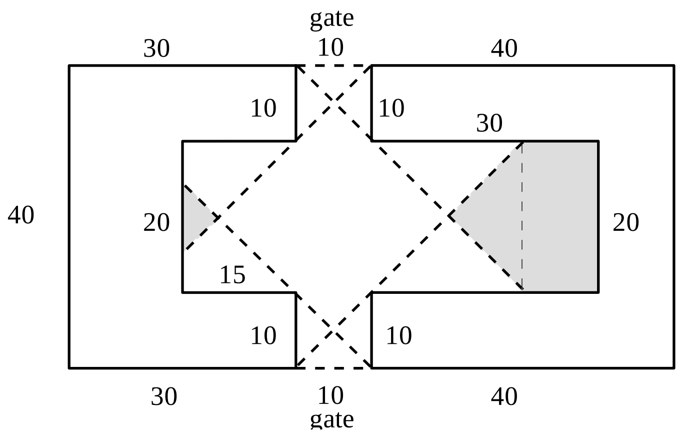
So there are $1393 - 511 = 882$ schedules which include all 3 activities.

M2.



This is the plan of a building which has a courtyard with two entrance gates. Passers-by can look in through the gates but may not enter. The dimensions are given in metres and all corners are right angles. What is the area of the part of the courtyard which cannot be seen by passers-by?

Solution



The two shaded areas are unseen.

Left hand unseen area:

Square with diagonal 10 m has side $\dfrac{10}{\sqrt{2}}$ m and area 50 m^2

So left hand unseen triangle has area 25 m^2.

Right hand unseen area:

Square with diagonal 20 m has side $\dfrac{20}{\sqrt{2}}$ m and area 200 m^2.

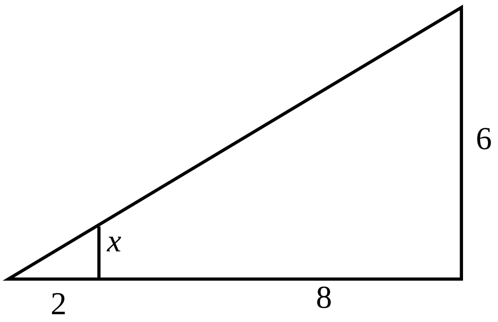
So right hand unseen triangle has area 100 m^2 and the right hand unseen rectangle has area 200 m^2.

Total unseen area is 325 m^2.

M3. When Max is 8 m from a lamp post which is 6 m high his shadow is 2 m long. When Max is 3 m from the lamp post, what is the length of his shadow?

Solution

Let Max be x m tall.



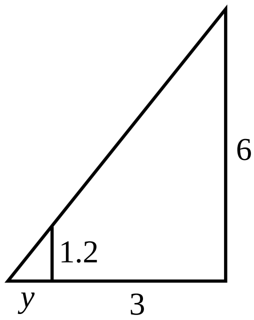
Then

$$\frac{6}{10} = \frac{x}{2}$$

so

$$x = \frac{6}{5} = 1.2.$$

Now when Max is 3 m from the lamp post let the length of his shadow be y m.



Then

$$\frac{6}{3 + y} = \frac{1.2}{y}$$

so

$$6y = 3.6 + 1.2y$$

$$4.8y = 3.6$$

$$y = \frac{3.6}{4.8} = 0.75$$

So when Max is 3 m from the lamp post the length of his shadow is 0.75 m or 75 cm.

M4. A parent has washed some nappies in a strong bleach solution and wishes to rinse them so that they contain as weak a bleach solution as possible. By wringing out, the nappies can be made to contain just half a litre of solution. Show that two thorough rinses, such that the solution strength is uniform, the first using 12 litres of water and the second using 8 litres of water, reduces the strength of the bleach solution to $\frac{1}{425}$ of its original value.

If 20 litres of clean water is all that is available and the parent is prepared to do only two rinses, how best should the water be divided between the two rinses?

Solution

In rinse 1, 12.5 litres of water with bleach is reduced to 0.5 litres, i.e. to $\frac{0.5}{12.5} = \frac{1}{25}$ of its original value.

In rinse 2, 8.5 litres of water with bleach is reduced to 0.5 litres, i.e. to $\frac{0.5}{8.5} = \frac{1}{17}$ of its original value.

So the two rinses in succession reduce to $\frac{1}{25} \times \frac{1}{17} = \frac{1}{425}$ of its original value.

Use x litres of water for the first rinse. In rinse 1 $(x + \frac{1}{2})$ litres of water with bleach is reduced to $\frac{1}{2}$ litre, i.e. to $\frac{\frac{1}{2}}{x + \frac{1}{2}} = \frac{1}{2x + 1}$ of its original value.

In rinse 2 $(20 - x + \frac{1}{2})$ litres of water with bleach is reduced to $\frac{1}{2}$ litre, i.e. to $\frac{\frac{1}{2}}{20 - x + \frac{1}{2}} = \frac{1}{41 - 2x}$ of its original value.

So the two rinses in succession reduce to $\frac{1}{(2x + 1)(41 - 2x)}$ of its original value.

Minimise this ratio by maximising

$$(2x + 1)(41 - 2x) = -4x^2 + 80x + 41$$
$$= -4(x - 10)^2 + 441.$$

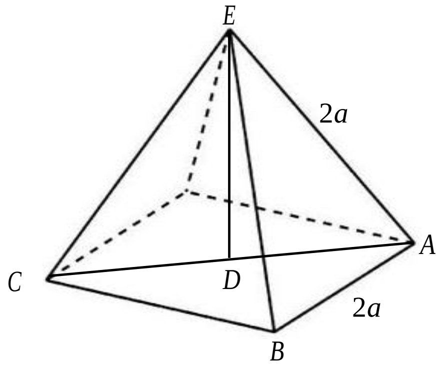
This is maximum when $x - 10 = 0$ i.e. $x = 10$.

So it is best to divide the water equally between the two rinses, which will reduce the bleach concentration to $\frac{1}{441}$ of its original value.

M5. A pyramid has a square base and four equilateral triangles as its other faces. The four equilateral triangles can also make a tetrahedron. What is the ratio of the volumes of the pyramid and the tetrahedron? **Justify your answer.**

Solution

Let the side length of the square be $2a$.
Therefore, the side length of the triangles is also $2a$.
Volume of a pyramid = $\frac{1}{3}$ × base area × height.



Area of the base = $4a^2$.
Consider the triangle ABC.

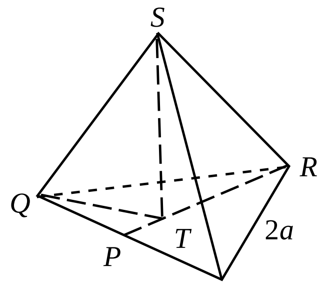
$$\begin{aligned} AC^2 &= (2a)^2 + (2a)^2 \\ &= 4a^2 + 4a^2 \\ &= 8a^2 \\ AC &= \sqrt{8a^2} = 2\sqrt{2}\,a \end{aligned}$$

Now consider triangle CDE, where $CD = \frac{1}{2}AC$.

$$\begin{aligned} DE^2 &= CE^2 - CD^2 \\ &= 4a^2 - 2a^2 \\ &= 2a^2 \\ DE &= \sqrt{2}\,a. \end{aligned}$$

Volume of the pyramid $= \frac{1}{3} \times$ base area $\times$ height $= \frac{1}{3} \times 4a^2 \times \sqrt{2}a = \frac{4\sqrt{2}}{3}a^3$.

Now consider the regular tetrahedron and triangle PQR.



$$PR^2 = QR^2 - PQ^2 = 4a^2 - a^2 = 3a^2$$

$$PR = \sqrt{3}a$$

Area of pyramid base = $\frac{1}{2} \times 2a \times \sqrt{3}a = \sqrt{3}a^2$

Now consider triangle PQT where $PT = \frac{1}{3}PR = \frac{1}{3}\sqrt{3}a$

$$\begin{aligned} QT^2 &= PQ^2 + PT^2 \\ &= a^2 + \frac{1}{3}a^2 = \frac{4}{3}a^2 \\ QT &= \frac{2\sqrt{3}}{3}a \end{aligned}$$

Consider triangle QST:

$$ST^2 = QS^2 - QT^2 = 4a^2 - \frac{4}{3}a^2 = \frac{8}{3}a^2$$

$$ST = \frac{2\sqrt{2}\sqrt{3}}{3}a.$$

Volume of the tetrahedron = $\frac{1}{3} \times \sqrt{3}a^2 \times \frac{2\sqrt{2}\sqrt{3}}{3}a = \frac{2\sqrt{2}}{3}a^3$.

$$\frac{\text{Volume of the pyramid}}{\text{Volume of the tetrahedron}} = \frac{\frac{4\sqrt{2}}{3}a^3}{\frac{2\sqrt{2}}{3}a^3} = 2.$$