Middle Division: Solutions to Problems 2

M1. How many positive multiples of 7 that are less than 1,000 end with the digit 3?
How many positive multiples of 7 that are less than 10,000 end with the digits 33?

Solution

If a multiple of 7 ends in a 3, the next multiple of 7 ends in a 0. As they also have to be multiples of 7, these are 70, 140, 210, ..., 980. There are 14 of these so there are 14 multiples of 7 less than 1000 which end with the digit 3.

If a multiple of 7 ends in 33, the next multiple of 7 ends in 40. These must also be such that when they are divided by 10 they produce a multiple of 7. These are 140, 840, 1540, 2240, 2940, 3640, 4340, 5040, 5740, 6440, 7140, 7840, 8540, 9240, 9940. So we can say there are 15 positive multiples of 7 that are less than 10 000 end with the digits 33. They are each of the numbers 140, 840, ... reduced by 7.

M2. A cross-country skier practices the same route from his home to his friend's home and leaves at the same time every day. He realised that when he skis at 10 mph he arrives at 4 minutes past noon, and when he skis at 15 mph he arrives at 4 minutes before noon. How fast would he have to go to reach his friend's house at noon exactly?

Solution

Let the distance be d miles. Then the time taken when skiing at 10mph is $\frac{d}{10}$ hours and the time taken when skiing at 15 mph is $\frac{d}{15}$ hours.

So, we have

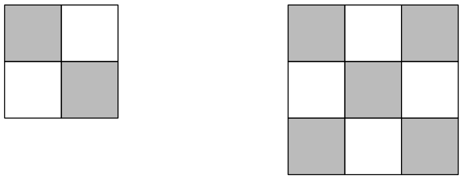
$$\begin{aligned} \frac{d}{10} &= \frac{d}{15} + \frac{8}{60} \\ 6d &= 4d + 8 \\ 2d &= 8 \\ d &= 4 \end{aligned}$$

The distance is 4 miles.

To cover 4 miles at 10 mph would take 0.4 of an hour which is 24 minutes. As he arrives at 12.04 skiing at 10 mph, it means he sets off at 11.40.

So to reach his friend's house at noon the time would be 20 minutes and
the speed would be $\frac{4}{\frac{1}{3}} = 4 \times \frac{3}{1} = 12$,
ie the skier will have to travel at 12 mph

M3. Consider an $n \times n$ grid of squares in which the squares alternate grey and white starting with grey in the top left hand corner. A 2×2 grid and a 3×3 grid will look like:



Using a grid of a given size, two black spots are placed on two different squares chosen at random.

For the two grids above (i.e. $n = 2, 3$), find out in how many distinct ways this can be done. What is the probability that both spots are on grey squares?

Obtain the corresponding results for the case $n = 9$.

Solution 1

$n = 2$: We can suppose the spots are placed one at a time. Now systematically go through all possible (i.e. three) positions for the first spot; e.g. start in the top left corner of the grid and work row by row: for the first position there are 3 other positions for the second spot; for the second position there are 2 others for the second spot, and for the third position, there is one other for the second spot. Thus there are $3 + 2 + 1 = 6$ ways the spots can be arranged. Alternatively, these 6 ways can be listed.
The spots are on the two grey squares in only one of the arrangements, so the probability that both spots are on grey squares is simply $\frac{1}{6}$.

$n = 3$: There are 9 squares in the grid, 5 of which are grey. In a similar way, there are $8 + 7 + \ldots + 1 = 36$ (a triangular number, with $36 = \frac{1}{2} \times 9 \times 8$) arrangements of the 2 spots.
The same argument can be used to count the number of arrangements in which the spots are only on grey squares; as there are 5 grey squares, there are $4 + 3 + 2 + 1 = 10$ (and $10 = \frac{1}{2} \times 5 \times 4$) arrangements.
The probability that both spots are on grey squares is then $\dfrac{10}{36} = \dfrac{5}{18}$.

$n = 9$: There are 81 squares in the grid, 41 of which are grey. Using the above arguments, the two spots can be arranged in $\frac{1}{2} \times 81 \times 80 = 3240$ ways, of which $\frac{1}{2} \times 41 \times 40 = 820$ are both on grey squares.
The probability that both spots are on grey squares is then $\dfrac{820}{3240} = \dfrac{41}{162}$.

Solution 2

$n = 2$: There are 4 possible squares for the first spot and 3 possible squares for the second spot, so the number of ways of placing the two spots is $4 \times 3 = 12$, assuming they look different.
But the spots are identical, so there are two ways of placing the two spots in the same pair of squares. So there are $\dfrac{4 \times 3}{2} = 6$ ways of placing two spots which look identical. But only one of these ways uses the two grey squares, so the probability that both spots are on grey squares is $\frac{1}{6}$.

$n = 3$: There are $\dfrac{9 \times 8}{2} = 36$ ways of placing two identical spots on a 3×3 grid. 5 of these squares are grey, so there are $\dfrac{5 \times 4}{2} = 10$ ways of placing the spots on grey squares. So the probability that both spots are on grey squares is $\dfrac{10}{36} = \dfrac{5}{18}$.

$n = 9$: There are $\dfrac{81 \times 80}{2}$ ways of placing two identical spots on a 9×9 grid. 41 of these squares are grey, so there are $\dfrac{41 \times 40}{2}$ ways of placing the spots on grey squares. So the probability that both spots are on grey squares is $\dfrac{41 \times 40}{81 \times 80} = \dfrac{41}{162}$.

M4. A positive integer ends in the digit 4 and has the property that it becomes four times as large when the 4 is moved from the end and placed at the front. What is the smallest such number?

Solution

Let the number be N and have $n + 1$ digits. So $N = 10x + 4$ where x has n digits.
Moving the 4 from the end to the front gives the number $4 \times 10^n + x$. So we have the equation

$$4(10x + 4) = 4 \times 10^n + x.$$

Rearranging gives that

$$39x = 4\left(10^n - 4\right).$$

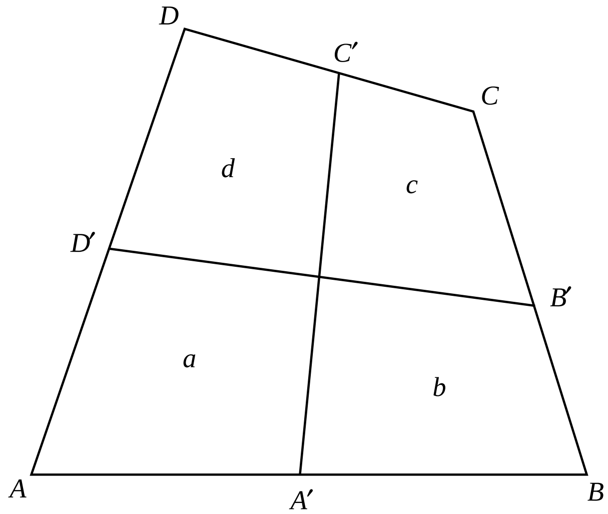
Thus 39 must divide $10^n - 4$.

Value of n	1	2	3	4	5
Value of $10^n - 4$	6	96	996	9996	99996

The smallest n for which this holds is $n = 5$ in which case $4\left(10^5 - 4\right) = 39 \times 10256$.
Check that $4 \times 102564 = 410256$. So the smallest such number is 102564.

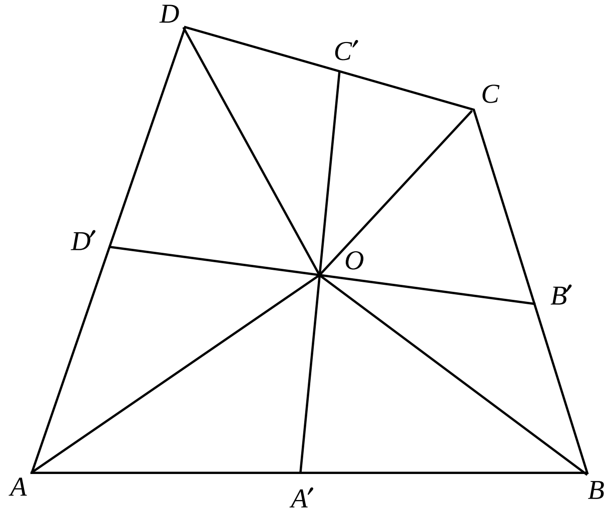
M5.

Let $ABCD$ be a quadrilateral. Let A' be the midpoint of AB, B' the mid-point of BC, C' the mid-point of CD and D' the mid-point of AD. Draw the lines $A'C'$ and $B'D'$ and let a, b, c, d be the areas of the four minor quadrilaterals as shown in the figure. Prove that $a + c = b + d$.



Solution

From the point of intersection O of the lines $A'C'$ and $B'D'$, draw the lines to each of the vertices of the quadrilateral. Recall that the area of a triangle is ½ × *base* × *height.*



Now triangles OAA' and OBA' have equal bases and the same height and so have the same area.
The same applies to the pairs of triangles (OBB', OCB'), (OCC', ODC') and (ODD', OAD').
So $a + c$ = area of triangle OAA' + area of triangle OAD' + area of triangle OCB' + area of triangle OCC' = area of triangle OBA' + area of triangle ODD' + area of triangle OBB' + area of triangle $ODC' = b + d$.