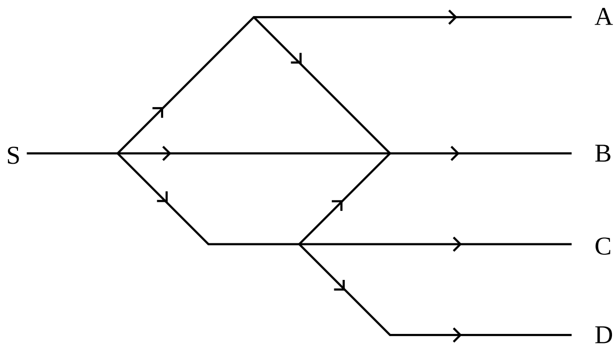
2019-2020 Middle Division: Problems 1 solutions

M1. Rhoda Rat is put in a maze at the start, S. She can move forward only in the direction of the arrows. At each junction she is equally likely to choose any of the forward paths.

What is the probability that she ends up at B?



Solution

At the three-way junctions Rhoda has a $\frac{1}{3}$ chance of taking each path and at the two-way junction she a $\frac{1}{2}$ chance of taking each path.

So her probability of arriving at B is

$$\left(\frac{1}{2} \text{ of } \frac{1}{3}\right) + \frac{1}{3} + \left(\frac{1}{3} \text{ of } \frac{1}{3}\right) = \frac{1}{6} + \frac{1}{3} + \frac{1}{9} = \frac{3 + 6 + 2}{18} = \frac{11}{18}.$$

M2. A contestant in a game show is offered three identical sealed boxes and asked to choose one.

He knows that one of the boxes contains £1000 and each of the other two contains a pebble.

The contestant chooses one of the boxes, but, before he opens it, the show host makes him an offer. "I know the contents of the three boxes, and will open one of the other two boxes to show you that there is a pebble inside it." He does this and then says, " If you wish, you can now change your mind about which of the boxes you want to open." The contestant is really keen to win the money so he says he will stick with his original choice of box. Was he wise? **Explain your answer.**

Solution

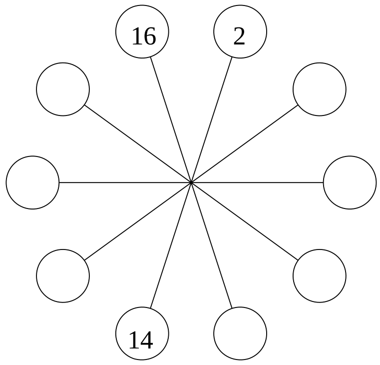
Label the boxes A, B and C and suppose that the contestant had chosen Box A.

The possible outcomes are:

Case 1.	Box A	Box B	Box C	
	Treasure	Pebble	Pebble	
				If he switches, he loses.
Case 2.	Pebble	Treasure	Pebble	
				If he switches, he wins.
Case 3.	Pebble	Pebble	Treasure	
				If he switches, he wins.

He would be better to change his choice.

M3. Place different integers in each of the remaining circles so that the sum of the squares of the two numbers in adjacent circles is equal to the sum of the squares of the numbers in the two diametrically opposite circles.



Solution

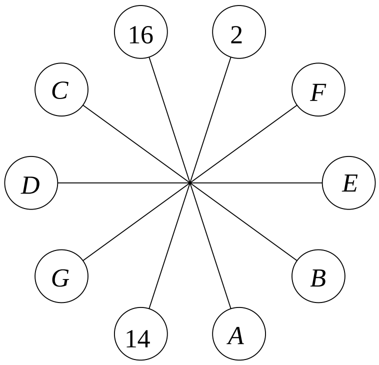


Figure 1

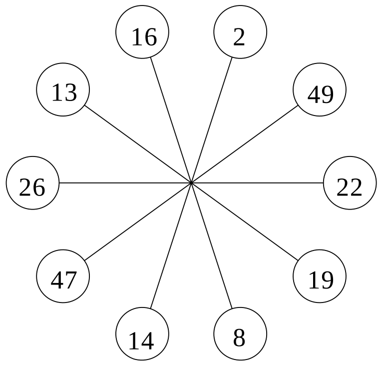


Figure 2

Let us label the circles as shown in Figure 1 and evaluate A. By the condition in the question

$$16^2 + 2^2 = 14^2 + A^2 \qquad A^2 = 256 - 196 + 4 \text{ so } A = 8.$$

Continuing anticlockwise we have

$$16^2 + C^2 = B^2 + 8^2 \qquad 192 = B^2 - C^2 = (B + C)(B - C).$$

Since $192 = 2 \times 2 \times 2 \times 2 \times 2 \times 2 \times 3$ we can create pairs for $B + C$ and $B - C$ as follows (where $B + C$ must exceed $B - C$).

$B - C$	$B + C$	B	C	
2	96	49	47	
3	64	$33\frac{1}{2}$	$30\frac{1}{2}$	not integers
4	48	26	22	
6	32	19	13	
8	24	16	8	used above
12	16	14	2	used above

Thus a total of 5 valid pairs are generated and these can be fitted in the diagram as shown **BUT** the order is not unique.

M4. There are 10 lockers in a row, numbered from 1 to 10. Each locker is to be painted red or blue or green, subject to the following rules:

- two lockers with numbers n and m are painted different colours whenever $n - m$ is odd
- it is not necessary to use all 3 colours;

In how many different ways can the row of lockers be painted? Justify your answer.

Solution

If the first locker is red, none of the even lockers can be red. And if another odd locker is green, none of the even lockers can be green. This means that all the even lockers must be blue. So we can paint all the even lockers in one colour and use either or both of the other two colours for the odd lockers, or vice versa.

Case 1

Just use two colours, alternating along the row. There are 3 ways of choosing the first colour and two ways of choosing the second colour, making $3 \times 2 = 6$ ways.

Case 2

Use one colour for the even lockers so 3 choices.

Use the other two colours for the odd lockers, making sure to use both colours. There are 2 choices for each of the 5 odd lockers, giving 25 = 32 possibilities. But this includes the two single colour possibilities, so there are $32 - 2 = 30$ ways of using both colours. So for the whole row there are $3 \times 30 = 90$ ways.

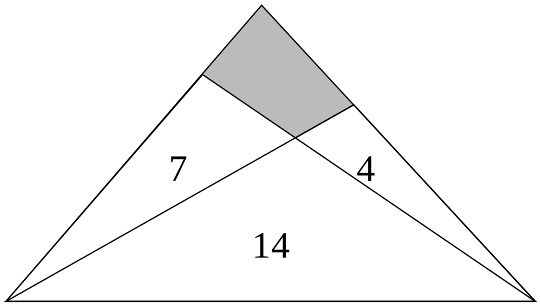
Case 3

Use one colour for the odd lockers and the other two colours for the even lockers, making sure to use both colours. Also 90 ways.

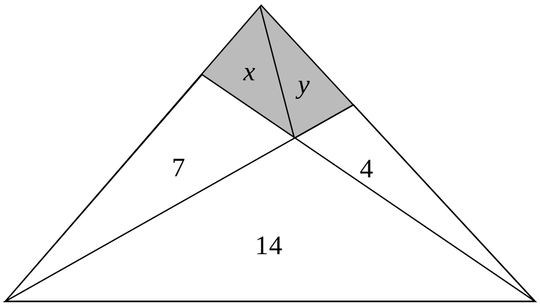
So there are $6 + 90 + 90 = 186$ different ways of painting the row of lockers.

M5. In the diagram (which is not drawn to scale) the small triangles each have the area shown.

Find the area of the shaded quadrilateral.



Solution



Draw in the line from the third vertex of the large triangle to the crossing point within it. Let the areas of the two small triangles formed be x and y as shown.

Triangles with the same height have areas proportional to the lengths of their bases. So

$$\frac{x}{7} = \frac{x+y+4}{21} \qquad x = \frac{1}{3}(x+y+4)$$

$$\frac{y}{4} = \frac{x+y+7}{18} \qquad y = \frac{2}{9}(x+y+7)$$

$$x + y = \left(\frac{1}{3} + \frac{2}{9}\right)(x+y) + \frac{4}{3} + \frac{14}{9}$$

$$\frac{4}{9}(x+y) = \frac{26}{9} \qquad x + y = \frac{26}{4} = \frac{13}{2}.$$

So the shaded area is $6\frac{1}{2}$.

NOTE: This method establishes the whole area but other methods will find x and y and add them.