2018-2019 Middle Division: Solutions to Problems 2

M1. An unlimited supply of petrol is available from a camp at one edge of a desert which is 800 miles wide but no petrol is available anywhere else. A truck can only carry enough petrol to travel 500 miles and is able to leave petrol to be collected later. (There is no limit on the size of such stocks and it should be assumed that no petrol is lost by evaporation or spillage.) Establish whether or not it is possible for the truck to get across the desert and, if it is, explain how.

Solution

Working backwards:

(a) With just 500 units of petrol, the lorry can go 500 miles.

(b) With 1000 units of petrol, the lorry fills up; goes $500 \div 3$ miles, leaves $500 \div 3$ units of petrol in the cache, returns, fills up, drives $500 \div 3$ miles then fills up and can go a further 500 miles. Total distance $500\left(1 + \frac{1}{3}\right)$ miles.

(c) With 1500 units of petrol, the lorry fills up, goes $500 \div 5$ miles, leaves $500 \times (3 \div 5)$ units of petrol, returns and repeats, so that $500 \times (6 \div 5)$ units of petrol have been left in the second cache. The the lorry fills up and returns to the second cache. The lorry uses $1 \div 5$ units of petrol to reach the second cache, so has $500 \times (4 \div 5)$ units of petrol left in the tank. So there are 500×2 (i.e. 1000) units of petrol in the second cache, ready to carry out step (b).

Total distance $500\left(1 + \frac{1}{3} + \frac{1}{5}\right) < 800$.

(d) We need to extend the range to 800 miles. As $800 = 500\left(1 + \frac{9}{15}\right)$, we need another 500/15 miles. We need to create a third cache of 1500 units of petrol ready for step (c).

The lorry fills up, drives 500/15 miles, leaves $500 \times \frac{13}{15}$ units of petrol, drives back and repeats twice. The third cache now contains $500 \times \frac{39}{15}$ units of petrol. Now the lorry sets off with $500 \times \frac{7}{15}$ units of petrol and drives to the third cache. Here the lorry has $500 \times \frac{7}{15}$ units of petrol in the tank, added to the $500 \times \frac{39}{15}$ units of petrol already there makes 1500 units of petrol, ready to carry out step (c).

Total distance $500\left(1 + \frac{1}{3} + \frac{1}{5} + \frac{1}{15}\right) = 500\left(1 + \frac{9}{15}\right) = 800$ miles as required.

Alternatively, the third cache could follow the pattern above at $500 \div 7$ miles. Then the lorry would arrive with some petrol left in the tank.

M2. Two ships, one 200 metres in length and the other 100 metres in length, travel at constant but different speeds. When travelling in opposite directions, it takes 20 seconds for them to completely pass each other. When travelling in the same direction, it takes 50 seconds for them to completely pass each other.

Find the speed of the faster ship.

Solution

Let the speed of the faster ship be v metres per second and the speed of the slower ship be u metres per second.

To pass, the faster ship must travel the length of the slower ship plus its own length, a total of

$$100 + 200 = 300 \text{ metres.}$$

When the ships travel in the same direction the relative speed is $v - u$ metres per second. So

$$v - u = \frac{300}{50} = 6.$$

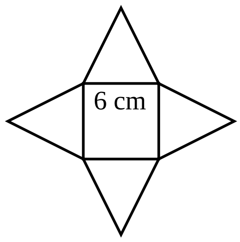
When the ships travel in opposite directions the relative speed is $v + u$ metres per second. So

$$v + u = \frac{300}{20} = 15.$$

Hence $2v = 21$ so $v = 10.5$.

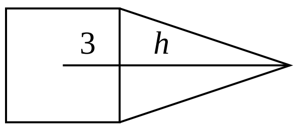
The speed of the faster ship is 10.5 metres per second.

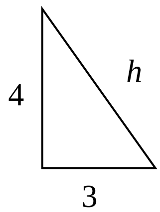
M3. Four identical isosceles triangles border a square of side 6 cm, as shown. When the four triangles are folded up they meet at a point to form a pyramid with a square base. If the height of this pyramid is 4 cm, find the total area of the four triangles and the square.



Solution

Let the height of the isosceles triangles be h cm.





The top of the pyramid lies directly above the centre of the square, which is 3 cm from the foot of h. So by Pythagoras' Theorem

$$3^2 + 4^2 = h^2$$

and hence $h = 5$.

The area of each of the four isosceles triangle is $\frac{1}{2} \times 6 \times 5 = 15$ cm^2.

The area of the square is $6 \times 6 = 36$ cm^2.

Hence the total area is $4 \times 15 + 36 = 96$ cm^2.

M4. Show that the product of four consecutive odd integers is always 16 less than a square number.

Deduce that the product of four consecutive odd integers can never be a square number except in one particular case.

Solution

Suppose the four integers are $k - 3$, $k - 1$, $k + 1$, $k + 3$ for some even integer k. Their product can be written

$$(k - 1)(k + 1)(k - 3)(k + 3)$$

$$= \left(k^2 - 1\right)\left(k^2 - 9\right) = k^4 - 10k^2 + 9 = \left(k^2 - 5\right)^2 - 16 \text{ (by completing the square).}$$

This proves the first part.

We inspect the sequence of squares 0, 1, 4, 9, 16, 25, 36, 49, 64, 81, 100,…

There are 2 pairs that differ by 16: 0 and 16, and 9 and 25. There cannot be any more as the difference between further successive squares is greater than 16.

0 cannot be a product of odd integers.

So the only situation where the product of four consecutive odd integers is a square is when the product is 9 and the odd integers are $-3,\ -1,\ 1,\ 3$.

M5. A cardboard box manufacturer makes open-topped boxes which are cubes. Because of changes in the market, there are plans to double the volume of the boxes which are made. The regular supplier of raw cardboard offers a 37.5% discount on the price that was originally being charged. A new supplier offers a deal in which the manufacturer would be paying exactly the same for the raw material for his bigger boxes as was paid for the smaller boxes.

Which is the best deal for the manufacturer?

Solution

Let the length of a side of the original box be a metres. So its volume is a^3 m^3 and the amount of cardboard required is $5a^2$ m^2. If the regular supplier charged £x per square metre, the raw material for each box costs £$5a^2x$.

He now doubles the volume of his boxes to $2a^3$ m^3. If the length of the side is A metres then $A^3 = 2a^3$ so that $A = \sqrt[3]{2}a$. The amount of cardboard required for the bigger box is $5A^2 = 5\sqrt[3]{2^2}a^2$ m^2.

So the regular supplier charges $5\sqrt[3]{2^2}a^2 \times \frac{5}{8}x$ and the new supplier charges £$5a^2x$. Since $\sqrt[3]{2^2} \times \frac{5}{8} < 1$ he should stick with the original supplier.