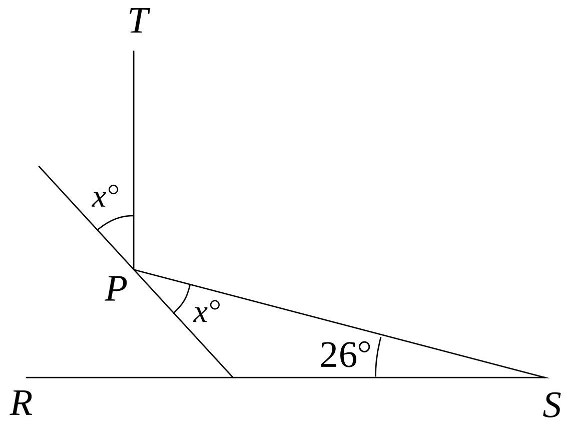
2018-2019 Middle Division: Problems 1 solutions

M1. A beam of light shines from point S, reflects off a reflector at point P, and reaches point T so that PT is perpendicular to RS and $\angle RSP = 26°$ as shown below. Find angle $x°$.



Solution 1

In the diagram, extend TP to meet RS at A. Since AT is perpendicular to RS

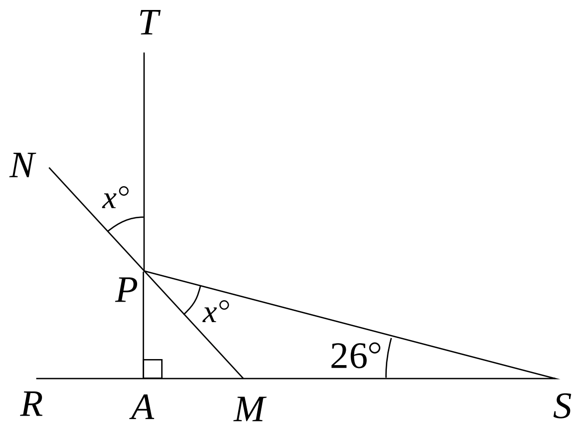
$$\angle SPA = 180° - 90° - 26° = 64°.$$

Label points M and N. Since $\angle TPN$ and $\angle MPA$ are vertically opposite angles, they are equal, so

$$\angle MPA = x°.$$

Since $\angle SPA = 2x° = 64°$, $x = 32$.

Thus, the angle $x°$ is $32°$.



Solution 2

In the diagram, draw the line PU parallel to RS.

$$\angle SPU = \angle RSP = 26° \text{ (alternate angles)}$$

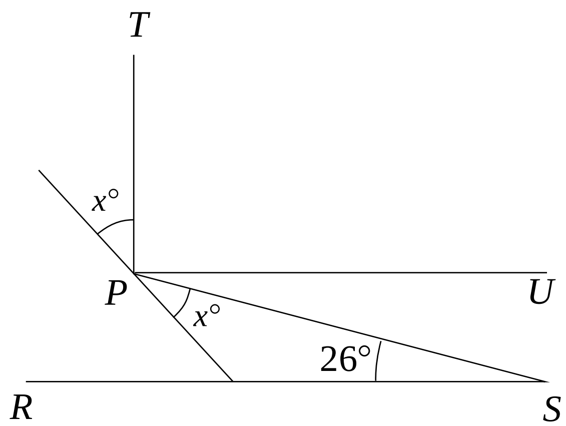
As PT is perpendicular to RS it is also perpendicular to PU so $\angle TPU = 90°$.

Thus, at the point P, we have

$$x° + 90° + 26° + x° = 180°$$

$$2x° = 180° - 90° - 26° = 64°.$$

Thus, the angle $x°$ is $32°$.



M2. Emma started with a rectangle of paper. With one straight cut she divided it into a rectangle and a square. She took the rectangle and with one straight cut divided it into a rectangle and a square, which was smaller than the previous one. She kept repeating this process until eventually the final rectangle was a square with sides 1 centimetre and she was left with a pile of squares of paper. The average area of the squares was a two digit number of square centimetres.

What were the dimensions of the original rectangle?

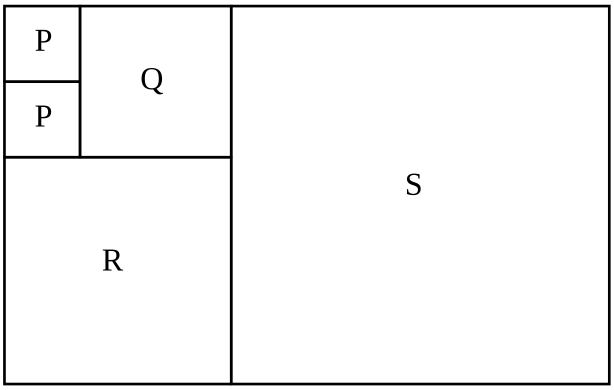
Solution

This solution reverses the dissection until the dimensions of the original paper are found.

Start with the two 1 cm squares (P) and put them together to make a rectangle with longer side 2 cm.
Add a 2 cm square (Q) to make the next rectangle with longer side 3 cm.

Add a 3 cm square (R) to make the next rectangle with longer side 5 cm.

Add a 5 cm square (S) to make the next rectangle with longer side 8 cm and so on.



Now analyse the dimensions. (Note that the sides of each square form a Fibonacci sequence where each term is the sum of the previous two terms.)

square side	area of largest square	total area	number of squares	average area
1	1	1	1	1
1	1	2	2	1
2	4	6	3	2
3	9	15	4	not an integer
5	25	40	5	8
8	64	104	6	not an integer
13	169	273	7	39
21	441	714	8	not an integer
34	1156	1860	9	over 100

So the original piece of paper was 21 cm by 13 cm.

M3. A party of 30 villagers decided to hire a bus to take them to a show in the city. The tickets for the show cost 50p for children, £2.50 for pensioners and £5 for others. The number of “others” attending was more than the number of children but less than twice the number of children. There were more children than pensioners on the bus.

The total cost of the tickets was £100. How many children and how many pensioners attended the show?

Solution

Let the numbers of children, pensioners and adults be c, p and a respectively. Then

$$c + p + a = 30; \qquad c < a < 2c, \qquad p < c.$$

Also, from the ticket money, $0.5c + 2.5p + 5a = 100$ which gives $c + 5p + 10a = 200$.
Since c, p and a are whole numbers, c must be a multiple of 5.

Note that c cannot be 0, since there are more children than pensioners.

If $c = 5$, then $5p + 10a = 195$ so $p + 2a = 39$… (1). Also, from $c + p + a = 30$, $p + a = 25$… (2). Subtracting (2) from (1) gives $a = 14$ and then $p = 11$. But we are told that there are more children than pensioners, so this is not possible. So $c \neq 5$.

If $c = 10$ then $5p + 10a = 190$ so $p + 2a = 38$… (3) and from $c + p + a = 30$, $p + a = 20$… (4). Subtracting (4) from (3) gives $a = 18$ and hence $p = 2$. This also satisfies the condition that $c < a < 2c$.

If $c = 15$ then $5p + 10a = 185$ so $p + 2a = 37$… (5) and from $c + p + a = 30$, $p + a = 15$… (6). Subtracting (6) from (5) gives $a = 22$ which gives $p = -7$ and a negative number is not possible!

Similarly, if $c > 15$ the value of p would still be negative.

So the only possibility is that 10 children and 2 pensioners attended the show.

M4. Three expert logicians played a game with a set of 21 cards each with a different two-digit prime number. Each drew a card and held it up so that they could not see the number on their own card but could see the number on the cards of each of the others. Ali, Bobby and Charlie in turn were then asked two questions, namely "Is your number the smallest of the three?" and "Is your number the largest of the three?". In the first round all three answered "Don't know" to both questions. The same happened in rounds two and three. In round 4 Ali answered "Don't know" to the first question. What did Ali answer to the second question and what numbers did Bobby and Charlie have?

Solution

There are exactly 21 two-digit primes:

11	13	17	19	23	29	31	37	41	43	
47	53	59	61	67	71	73	79	83	89	97

It is only the order which is important, so consider the numbers 1, 2, ... 21 instead.

Round 1

If A could see the 1 card, she would know she had a number greater than 1 and would answer no to the first question. So B and C cannot have 1.

If B could see the 1 card or the 2 card, he would know he had a number greater than 2 and would answer no to his first question. So A cannot have 1 or 2 and C cannot have 2.

If C could see the 2 card or the 3 card, she would know she had a number more than 3 and would answer no to the first question. So A cannot have 3 and B cannot have 2 or 3.

If A could see the 21 card, she would know she had a number less than 21 and would answer no to the second question. So B and C cannot have 21.

If B could see the 21 card or the 20 card, he would know he had a number less than 20 and would answer no to his second question. So A cannot have 21 or 20 and C cannot have 20.

If C could see the 20 card or the 19 card, she would know she had a number less than 19 and would answer no to the second question. So A cannot have 19 and B cannot have 20 or 19.

Rounds 2 and 3 are similar. The numbers eliminated each time are:

round	question	A	B	C
1	A1		1	1
	B1	1, 2		2
	C1	3	2, 3	
2	A1		4	3, 4
	B1	4, 5		5
	C1	6	5, 6	
3	A1		7	6, 7
	B1	7, 8		8
	C1	9	8, 9	
1	A2		21	21
	B2	21, 20		20
	C2	19	20, 19	
2	A2		18	19, 18
	B2	18, 17		17
	C2	16	17,16	
3	A2		15	16, 15
	B2	15, 14		14
	C2	13	14,13	

So A can only have 10, 11 or 12, B can only have 10, 11 or 12 and C can only have 9, 10, 11, 12 or 13.

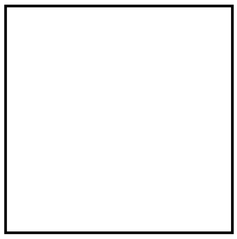
A can't see 9 or 10, because then she would know to answer no to the next question. And she can't see both 11 and 12 or both 12 and 13 because then she would answer yes.

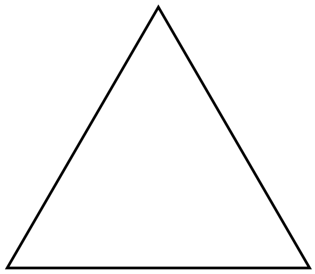
So she must be able to see 11 and 13.

Hence Ali would answer no to the question "Is your number the largest of the three?". And Bobby has the 11th prime, 47 and Charlie has the 13th prime, 59.

M5. Consider a square with side 15 cm and an equilateral triangle with the same perimeter. Which has the greater area? And by how much?

Solution

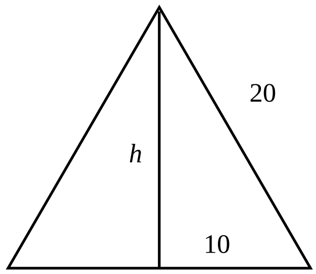




The area of the square is $15\text{ cm} \times 15\text{ cm} = 225\text{ cm}^2$ and its perimeter is $4 \times 15\text{ cm} = 60\text{ cm}$.

So each side of the triangle is 20 cm. We will calculate the area of the triangle in three ways:

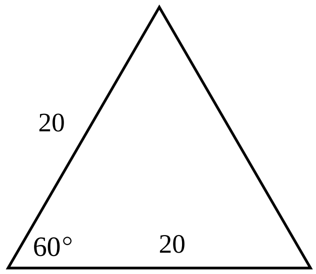
Method 1:
Put in an altitude. This bisects the base and we can use Pythagoras' Theorem:



$$h^2 + 10^2 = 20^2$$
$$h^2 = 400 - 100 = 300$$
$$h = \sqrt{300}$$

So the area is $\frac{1}{2} \times 20 \times \sqrt{300} = 100\sqrt{3}\text{ cm}^2$.

Method 2:
We can use trigonometry. The area is given by



$$\tfrac{1}{2} \times 20 \times 20 \times \sin 60^\circ$$
$$= 10 \times 20 \times \frac{\sqrt{3}}{2}$$
$$= 100\sqrt{3}\text{ cm}^2$$

Method 3:
We can use Heron's Formula which states that the area of a triangle is given by

$$\sqrt{s(s-a)(s-b)(s-c)}$$

where the sides are a, b, c and s is the semi-perimeter, $\frac{1}{2}(a+b+c)$. So the area is

$$\sqrt{30 \times 10 \times 10 \times 10} = \sqrt{3 \times 100 \times 100} = 100\sqrt{3}\text{ cm}^2$$

Thus the area of the square is 225 cm^2, the area of the triangle is $100\sqrt{3}\text{ cm}^2$ and the area of the square is bigger since

$$100\sqrt{3} < 100\sqrt{4} = 200 < 225.$$

And the area of the square is bigger than the area of the triangle by

$$225 - 100\sqrt{3} = 25\left(9 - 4\sqrt{3}\right) \approx 25 \times \left(9 - 6{\cdot}9\right) \approx 50\text{ cm}^2.$$