Middle Division 2017-2018 Round 2 Solutions

M1. A strange announcement was made on the radio about a local election with three candidates: Mrs Allan, Mr Baxter and Ms Campbell.

"Mrs Allan beat Mr Baxter by one eighth of the total votes cast.

Mr Baxter beat Ms Campbell by a seventh of the total votes cast.

The votes cast for Mrs Allan was 350 fewer than 3 times Ms Campbell's votes."

How many votes did each candidate get?

Solution

Let a, b, c be the number of votes for Mrs Allan, Mr Baxter and Ms Campbell respectively.

$$a = b + \tfrac{1}{8}(a+b+c) \qquad b = c + \tfrac{1}{7}(a+b+c) \qquad a = 3c - 350 \ldots (3)$$

$$8a = 8b + a + b + c \qquad 7b = 7c + a + b + c$$

$$7a = 9b + c \ldots (1) \qquad 6b = 8c + a \ldots (2)$$

Substitute (2) into (1)

$$\begin{aligned} 7a &= 12c + \tfrac{3}{2}a + c \\ 11a &= 26c \\ a &= \frac{26}{11}c \end{aligned}$$

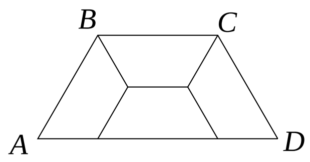
Substitute into (3)

$$\begin{aligned} \frac{26}{11}c &= 3c - 350 \\ 26c &= 33c - 3850 \\ 7c &= 3850 \\ c &= 550 \end{aligned}$$

$$\text{So,} \quad a = 1300 \quad \text{and} \quad b = 950$$

Mrs Allan got 1300 votes, Mr Baxter got 950 votes and Ms Campbell got 550 votes.

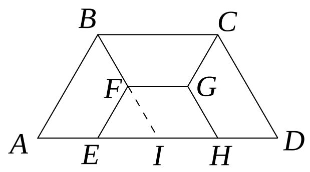
M2. A trapezium *ABCD* is split into four identical trapezia as shown below.



Given that *AB* has length 6 cm, find the area of *ABCD*.

Solution

We insert extra labels and line as shown below.



The trapezia are identical so $AB = BC = CD = EH$ and each is 6 cm.
All of the short sides of the trapezia are equal i.e. $EF = FG = GH = AE = BF = GC = DH$.
Since $FGHI$ is a parallelogram (actually a rhombus) $FI = GH = BF$ and as $BCDI$ is also a parallelogram $BI = CD$ hence $BF = \frac{1}{2}BI = 3$ cm as do all the other short lengths.
So $AD = 3 + 6 + 3 = 12$ cm

Consider triangle ABE: $BE^2 = 6^2 - 3^2 = 36 - 9 = 27$.
The area of a trapezium is half of the sum of the parallel sides multiplied by the distance between them.
Area of $ABCD = \frac{1}{2}(12 + 6) \times 3\sqrt{3} = 27\sqrt{3}$ cm^2 .

M3. Red, yellow and blue counters are placed on a board as shown, and they 'race' to the finish (F) by moving up one square at a time. The moves are determined by picking a bead at random from a bag containing one red bead, two yellow beads and three blue beads. After the colour of the bead which has been drawn is noted, the bead is returned to the bag before the next bead is picked. The race is over as soon as one of the counters lands on the square marked F. Find the probability of winning for each of the counters.

F	F	F
R		
	Y	
		B

Solution

The ways in which blue can win are (with an obvious notation)
BBB with probability $\left(\frac{3}{6}\right)^3 = \frac{1}{8}$
BBYB BYBB YBBB with probability $3 \times \frac{2}{6}\left(\frac{3}{6}\right)^3 = \frac{1}{8}$.
So P(blue wins) = $2 \times \frac{1}{8} = \frac{1}{4}$

The ways in which yellow can win are
YY with probability $\left(\frac{2}{6}\right)^2 = \frac{1}{9}$
BYY YBY with probability $2 \times \frac{3}{6}\left(\frac{2}{6}\right)^2 = \frac{1}{9}$
BBYY BYBY YBBY with probability $3 \times \left(\frac{3}{6}\right)^2\left(\frac{2}{6}\right)^2 = \frac{1}{12}$
So P(yellow wins) = $2 \times \frac{1}{9} + \frac{1}{12} = \frac{11}{36}$.

Finally P(red wins) = 1 – P(blue wins) – P(yellow wins) = $1 - \frac{1}{4} - \frac{11}{36} = \frac{16}{36} = \frac{4}{9}$.

Answer is: $\frac{1}{4}$ for Blue, $\frac{11}{36}$ for Yellow and $\frac{4}{9}$ for Red.

M4. Goliath and David play a game in which there are no ties. Each player is equally likely to win each game. The first player to win 4 games becomes the champion, and no further games are played. Goliath wins the first two games. What is the probability that David becomes the champion?

Solution

Since the first player to win 4 games becomes the champion, Goliath and David play at most 7 games. (The maximum number of games comes when the two players have each won 3 games and then one player becomes the champion on the next (7th) game.) We are told that Goliath wins the first two games.

For David to become the champion, the two players must thus play 6 or 7 games, because David wins 4 games and loses at least 2 games. We note that David cannot lose 4 games, otherwise Goliath would become the champion.

If David wins and the two players play a total of 6 games, then the sequence of wins must be GGDDDD. (Here D stands for a win by David and G stands for a win by Goliath.)

If David wins and the two players play a total of 7 games, then David wins 4 of the last 5 games and must win the last (7th) game since he is the champion.

Therefore, the sequence of wins must be GGGDDDD or GGDGDDD or GGDDGDD or GGDDDGD. (In other words, Goliath can win the 3rd, 4th, 5th, or 6th game.)

The probability of the sequence GGDDDD occurring after Goliath has won the first 2 games is

$$\left(\frac{1}{2}\right)^4 = \frac{1}{16}.$$

This is because the probability of a specific outcome in any specific game is $\frac{1}{2}$, because each player is equally likely to win each game, and there are 4 games with undetermined outcome.

Similarly, the probability of each of the sequences GGGDDDD, GGDGDDD, GGDDGDD, and GGDDDGD occurring is

$$\left(\frac{1}{2}\right)^5 = \frac{1}{32}.$$

Therefore, the probability that Goliath wins the first two games and then David becomes the champion is

$$\frac{1}{16} + 4 \times \frac{1}{32} = \frac{3}{16}.$$

M5. Let n be a three-digit number and let m be the number obtained by reversing the order of the digits in n. Suppose that m does not equal n and that $n + m$ and $n - m$ are both divisible by 7. Find all such pairs n and m.

Solution

Since $(n + m) + (n - m) = 2n$ and $(n + m) - (n - m) = 2m$, 7 divides both $2n$ and $2m$; hence 7 divides both n and m. Let $n = 100a + 10b + c$; then $m = 100c + 10b + a$.

We can assume, by interchanging n and m if necessary, that $a > c$ (noting that $a \neq c$). Since $100 = 7 \times 14 + 2$ and $10 = 7 + 3$, we have

$$n = (14 \times 7 + 2)\,a + (7 + 3)\,b + c$$

$$= 7 \times (14a + b) + (2a + 3b + c),$$

and, since n is divisible by 7, so is $2a + 3b + c$. Similarly, $2c + 3b + a$ is also divisible by 7. Subtracting these,

$$(2a + 3b + c) - (2c + 3b + a) = a - c,$$

so we deduce that $a - c$ is divisible by 7. Since a and c are integers between 0 and 9 and, by arrangement, $a > c$, $a - c = 7$. There are only three possibilities:

either $a = 8$ and $c = 1$; or $a = 9$ and $c = 2$; or $a = 7$ and $c = 0$.

Since $2a + 3b + c$ is divisible by 7, the first case gives $3b + 17$ is divisible by 7 and hence $3b + 3 = 3(b + 1)$ is as well, giving $b = 6$. In the second case, $3b + 20$ is divisible by 7 and hence $3b - 1$ is as well, giving $b = 5$. In the third case, b must be divisible by 7, so $b = 0$ or $b = 7$.

Thus the only possible pairs of numbers are $\{861,\ 168\}$, $\{952,\ 259\}$, $\{700,\ 007\}$ or $\{770,\ 077\}$.