Middle Division 2017-2018 Round 1 Solutions

M1. The value of 50! is the product of all the whole numbers from 1 to 50 inclusive, i.e.

$$50! = 1 \times 2 \times 3 \times 4 \times \ldots \times 49 \times 50.$$

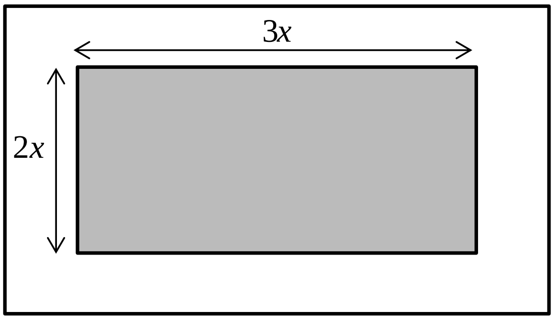
Find how many times 2 will divide 50!.

Solution

There are 25 even numbers which are less than or equal to 50 so 25 factors are divisible by 2.
12 of these factors are divisible by 4 so divisible by a second factor of 2
6 of these factors are divisible by 8 so divisible by a third factor of 2
3 of these factors are divisible by 16 so divisible by a fourth factor of 2
Finally, 1 of these factors is divisible by 32 so divisible by a fifth factor of 2
There are no factors are divisible by 64 or more.
So 2 will divide 50! $25 + 12 + 6 + 3 + 1 = 47$ times.

M2. A path, 3 metres wide, runs around the outside edge of a rectangular court.
The court is half as long again as it is wide.
The area of the path is 1596 square metres.
What are the dimensions of the court?

Solution



Let the length and breadth of the court be $3x$ metres and $2x$ metres.
As the width of the path is 3 metres, the overall dimensions are $3x + 6$ metres and $2x + 6$ metres.
Area of path

$$\begin{aligned}(3x + 6)(2x + 6) - 3x \times 2x &= 1596\\ 12x + 18x + 36 &= 1596\\ 30x &= 1560\\ x &= 52\end{aligned}$$

Dimensions of rectangle are 104 metres by 156 metres.

M3. How many times must I toss a coin in order that the odds are more than 100 to 1 that I get at least one head?

Solution

P (tail) = $\frac{1}{2}$

After n tosses, P (all tails) = $\left(\frac{1}{2}\right)^n$

P (at least one head) = $1 - \left(\frac{1}{2}\right)^n$

So, $$1 - \left(\frac{1}{2}\right)^n > \frac{100}{101}$$

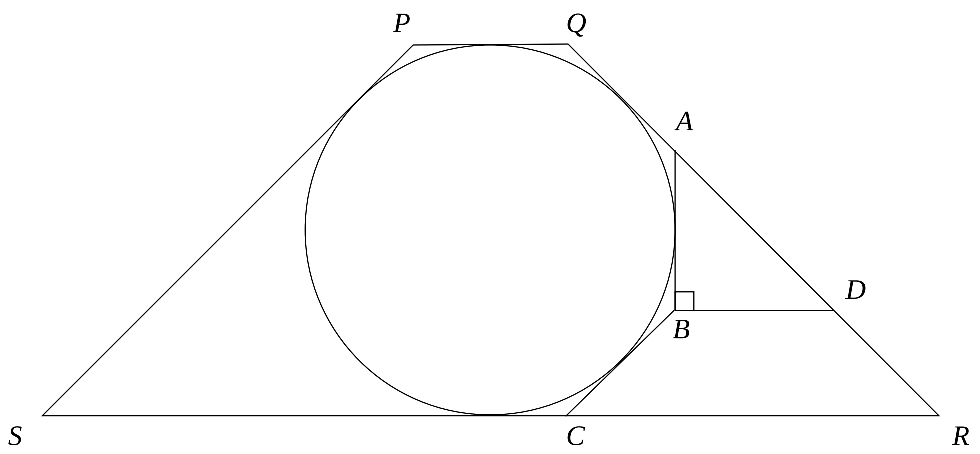
$$\left(\frac{1}{2}\right)^n < \frac{1}{101}$$

$2^n > 101$

$2^6 = 64$ and $2^7 = 128$ so $n = 7$

M4. In a trapezium $PQRS$, PQ is parallel to SR and $\angle SPQ = \angle RQP = 135°$.
The trapezium contains an inscribed circle and the length of PQ is 1 cm.
What is the **exact** length of QR?

Solution 1



A regular octagon contains an inscribed circle which must touch all sides. The angle between adjacent sides of a regular octagon is 135° and all pairs of opposite sides are parallel.

The trapezium $PQRS$ has an inscribed circle touching all its sides. The angle between adjacent sides, SP and PQ, is 135°. Similarly for $\angle PQR$ between PQ and SR. The opposite sides PQ and QR are given as parallel. Thus the regular octagon with one side PQ and the same inscribed circle as the trapezium $PQRS$ has parts of its other sides coincident with SP, QR and SR as shown in the diagram.

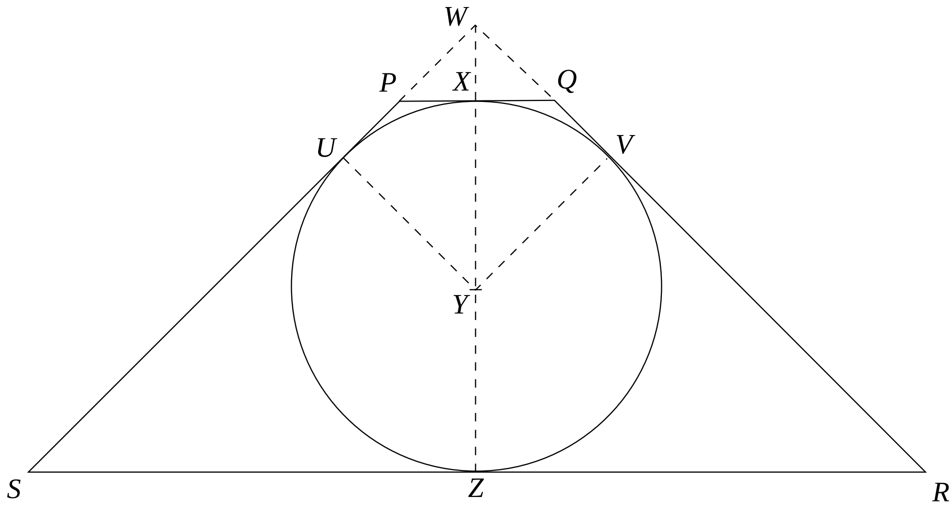
The lines QA, AB and BC are also sides of the regular octagon.

BD is parallel to SR.

$AB = BD = 1$ cm, so $AD = \sqrt{2}$ cm.

So $QR = QA + AD + DR = \left(1 + \sqrt{2} + 1\right)$ cm $= \left(2 + \sqrt{2}\right)$ cm.

M4. *Solution* 2



Extend the sides SP and RQ to meet at W and let X be the midpoint of PQ, Y the centre of the circle and Z the foot of the vertical diameter.

Since $\angle SPQ = 135°$, we know that $\angle QPW = 45°$ and similarly $\angle WQP = 45°$ so $\angle PWQ = 90°$ and WPQ is an isosceles right-angled triangle in which $WP = WQ$ so

$$WP^2 + WQ^2 = PQ^2$$

$$2WP^2 = 1^2 \Rightarrow WP^2 = \frac{1}{2} \Rightarrow WP = \frac{1}{\sqrt{2}}.$$

But X is the midpoint of PQ so $PX = \frac{1}{2}$ and $QX = \frac{1}{2}$.

Now draw in the radii YU and YV. Since SW and RW are tangents $\angle YUW$ and YVW are both 90° so $WVYU$ is a square. Furthermore, since they are equal tangents, $UP = PX = \frac{1}{2}$ and $VQ = QX = \frac{1}{2}$.

So, we have $UW = UP + PW = \frac{1}{2} + \frac{1}{\sqrt{2}} = \frac{1+\sqrt{2}}{2}$. But $WVYU$ is a square so the radii YU and YV are both equal to UW. So we have

$$WZ = WX + XY + YZ = \frac{1}{2} + \left(\frac{1}{2} + \frac{1}{\sqrt{2}}\right) + \left(\frac{1}{2} + \frac{1}{\sqrt{2}}\right) = \frac{3 + 2\sqrt{2}}{2}$$

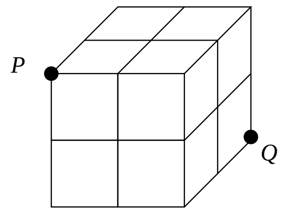
As PQ is parallel to SR we now know that $\angle ZRQ = 45°$ and, from above, $\angle XWQ = 45°$ so triangle WZR is isosceles hence

$$WZ^2 + ZR^2 = WR^2 \Rightarrow WR^2 = 2\left(\frac{3 + 2\sqrt{2}}{2}\right)^2 = \left(\frac{3 + 2\sqrt{2}}{\sqrt{2}}\right)^2$$

Hence

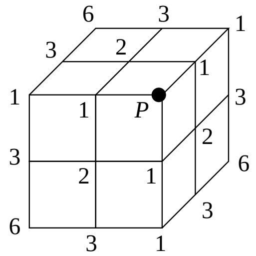
$$QR = WR - WQ = \frac{3}{\sqrt{2}} + 2 - \frac{1}{\sqrt{2}} = \sqrt{2} + 2.$$

M5. Each of the six faces of a solid cube is divided into four squares as indicated in the diagram. Starting from vertex P paths can be travelled to vertex Q along connected line segments. Each movement along a path must take one closer to Q. How many possible paths are there from P to Q?

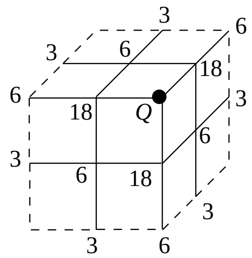


Solution

First count the paths from P towards the edges visible in the diagram below. At each node, the number of paths is the sum of the number of paths to the previous possible nodes.



Then look at the other side of the cube with Q in the centre and continue. Note that the dashed lines have already been traversed.



Thus there are 54 paths from P to Q.