Middle Division 2016-2017 Round 1 Solutions

M1. A victorious football team in an open-top bus is scheduled to leave the home ground and arrive at the town hall at 11 am. If the bus travels at 15 mph it will arrive 8 minutes early. However if it travels at 10 mph it will arrive 8 minutes late. At what speed must it travel to arrive at 11 am exactly?

Solution

Let the distance be d miles and the required travel time t hours. Then

$$\frac{d}{15} = t - \frac{8}{60}$$

$$\frac{d}{10} = t + \frac{8}{60}$$

Adding

$$\frac{d}{15} + \frac{d}{10} = 2t$$

$$\frac{d}{12} = t$$

So the required speed is 12 mph.

M2. (a) Adam has a five-digit number

$$* * * * *$$

When he places a 1 at the end of this number it becomes three times larger than when he placed a 1 at the start.

Find the five-digit number.

(b) If you added a 1 in the same way to a 3-digit number how many times larger would it be?

Solution

(a)

$$* * * * * 1 = 3 \times 1 * * * * *$$

Let the five digit number be x.

$$10x + 1 = 3(100\,000 + x)$$

$$10x + 1 = 300\,000 + 3x$$

$$7x = 299\,999$$

$$x = 42\,857$$

(b)

Three digit number

$$* * * 1 = n \times 1 * * *$$

Let the three digit number be y.

$$10y + 1 = n(1000 + y)$$

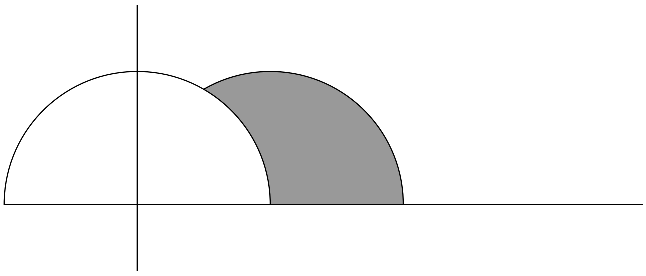
$$(10 - n)y = 1000n - 1$$

List the possibilities for $n = 1$ to 9 and the only ones which give y as an integer are $n = 1$, 7 and 9 but $n = 1$ means that the value has not changed. In this case $10y + 1 = 1000 + y$, $9y = 999$. So $y = 111$.

However, $n = 7$ or 9 both lead to y as a four-digit number (2333 or 8999).

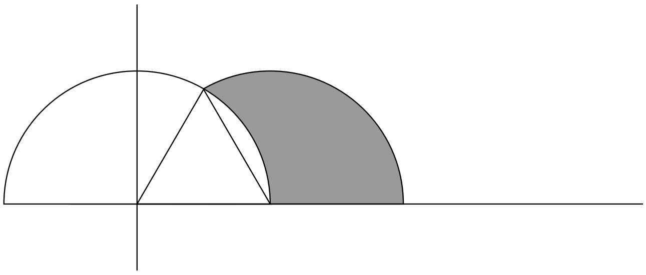
So this only works for a three-digit number when the number is 1 times as large i.e. unchanged.

M3.



The diagram shows parts of two circles, each with radius 1 unit, and centres on the *x*-axis at $x = 0$ and $x = 1$. Determine the **exact** value of the shaded area.

Solution



Shaded area

= 1/3 circle sector - 1/6 circle segment

= $\pi/3$ - (1/6 circle sector - equilateral triangle side 1)

= $\pi/3 - (\pi/6 - \sqrt{3}/4)$

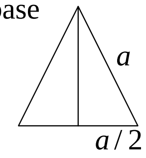
= $\pi/6 + \sqrt{3}/4$

M4. A pyramid stands on horizontal ground. Its base is an equilateral triangle with sides of length a, the other three edges of the pyramid are of length b and its volume is V. Show that the volume of the pyramid is

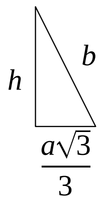
$$V = \frac{1}{12}a^2\sqrt{3b^2 - a^2}.$$

Solution

base



vertical section through corner



Area of base of pyramid = $\frac{1}{2}a\sqrt{3}\,\frac{a}{2} = a^2\frac{\sqrt{3}}{4}$

The centre of the base of the pyramid is $\frac{2}{3}\sqrt{3}\,\frac{a}{2} = \frac{a\sqrt{3}}{3}$ from a corner.

Let the height of the pyramid be h.

$$h^2 = b^2 - \left(\frac{a\sqrt{3}}{3}\right)^2 = b^2 - \frac{a^2}{3}$$

$$h = \sqrt{\frac{3b^2 - a^2}{3}}$$

Hence

$$V = \frac{1}{3} \times \text{ area of base} \times \text{ height}$$

$$= \frac{1}{3}a^2\frac{\sqrt{3}}{4}\sqrt{\frac{3b^2 - a^2}{3}}$$

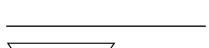
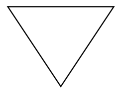
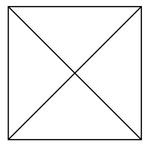
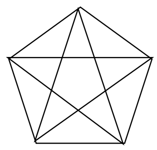
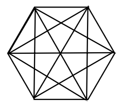
$$= \frac{1}{12}a^2\sqrt{3b^2 - a^2}$$

M5. Eight islands each have one or more air services. An air service consists of flights to and from another island, and no two services link the same pair of islands. There are 17 air services in all between the islands.

Show that it must be possible to use these air services to fly between any pair of islands.

Solution

If it is not possible then the air services divide the islands into two or more disconnected groups. The maximum number of services is when there are two groups − otherwise additional services could be added to reduce the number of groups to two. Every island has at least one service so the the smallest group size is 2, and the possible sizes of the two groups are 2, 6; 3, 5; and 4, 4. The maximum number of services within a group is when every island in the group is linked directly to every other island in the group:

group sizes	max. number of services within groups	
2	1	
3	3	
4	6	
5	10	
6	15	

group sizes	max. number of services within groups
2 6	1 + 15 = 16
3 5	3 + 10 = 13
4 4	6 + 6 = 12

Hence there can be at most 16 services when the islands are in two disconnected groups. So the 17th service must link the two groups and make it possible to fly between any pair of islands.