2015-2016 Middle Solutions Round 2

M1 A pairs jousting tournament in which each knight would fight every other knight in the competition (unless he had to withdraw due to serious injury) was just about to start. Some unknown knights rode up and asked to be allowed to take part in the tournament. It was decided to include the unknown knights and 26 more pairs competitions had to be scheduled.

How many knights were taking part originally and how many unknown knights arrived?

Solutions

Suppose that there were s jousters at the start and an extra y knights rode up. Then $\frac{1}{2}s(s - 1)$ jousts were scheduled originally.

The new number of jousts required $= \frac{1}{2}(s + y)(s + y - 1)$.

$$\frac{1}{2}(s + y)(s + y - 1) - \frac{1}{2}s(s - 1) = 26 \Rightarrow y(2s + y - 1) = 52.$$

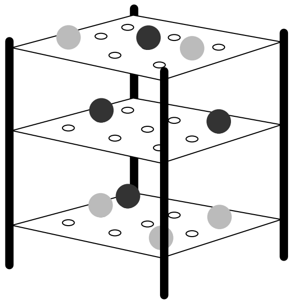
Now, y is a whole number and also a factor of 52; i.e. 1, 2, 4, 13, 26 or 52.
"Some" knights rode up so y is not 1.

If $y = 13$, 26, or 52, s is negative, so y is either 2 or 4.
If $y = 2$, then s is not a whole number; if $y = 4$, $s = 5$.
So originally there were 5 knights and another 4 appeared.

M2 Imagine a three-dimensional version of noughts and crosses: two players take it in turn to place different coloured marbles in a $3 \times 3 \times 3$ cube arrangement as shown in the diagram.

The object of the game is to create as many lines of three marbles of your own colour as possible.

How many different possible lines are there?



Solution 1

There are 8 lines within each of 3 layers of the large cube − total $8 \times 3 = 24$ lines.

Considering one vertical direction only, there are 5 lines not yet counted in each of 3 slices − $5 \times 3 = 15$ in total.

In the other vertical direction, there are 2 lines not yet counted in each of 3 slices - $2 \times 3 = 6$ in total.

Finally there are 4 lines between opposite corners of the large cube.

Hence the total number of lines is 49.

Solution 2

There are 7 lines through each of 8 vertices of the large cube. Each line has its other end at another vertex, and so the total number of these lines is $7 \times 8/2 = 28$.

There are 3 lines starting at the midpoint of each of 12 edges of the large cube. Each line has its other end at another midpoint, and so the total number of these lines is $3 \times 12/2 = 18$.

There is one line starting at the midpoint of each of 6 faces of the large cube. Each line has its other end at the midpoint of another face, and so the total number of these lines is $1 \times 6/2 = 3$.

Hence the total number of lines is 49.

M3. A circle of radius 15 cm intersects another circle of radius 20 cm at right angles.

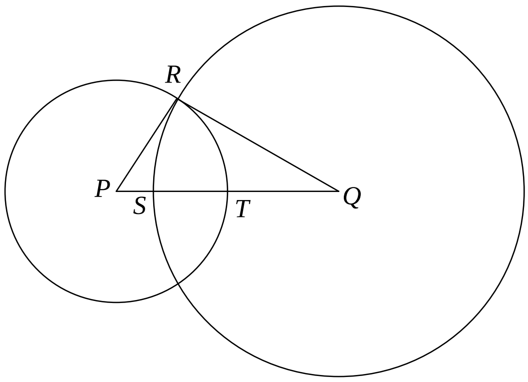
Work out an exact expression for the difference between the areas of the non-overlapping portions.

What is the sum of the areas of the non-overlapping portions?
Give your answer to four significant figures.

Solutions

On the diagram below, let X be the area of the overlapping portion. Then the difference in areas is

$$\left(20^2\pi - X\right) - \left(15^2\pi - X\right) = \pi\left(20^2 - 15^2\right) = 175\pi \text{ cm}^2.$$



The sum of the overlapping areas is

$$\left(20^2\pi - X\right) + \left(15^2\pi - X\right) = \pi\left(20^2 + 15^2\right) - 2X.$$

Now X is equal to the twice the area of the curved region RST on the diagram.

The area of $\triangle PQR$ = the area of the sector PRT + the area of the sector QSR − the area of RST.

Since the radii are 15 and 20 and we are told that $\angle PRQ = 90°$, $\tan \angle RPQ = \frac{20}{15}$ and $\tan \angle RQP = \frac{15}{20}$. So the area of RST is given by

$$\text{sector } RPT + \text{sector } RQS - \triangle PQR$$

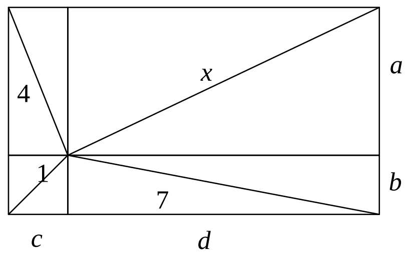
$$= 20^2\pi\left(\frac{\tan^{-1}\frac{15}{20}}{360°}\right) + 15^2\pi\left(\frac{\tan^{-1}\frac{20}{15}}{360°}\right) - \frac{1}{2} \times 15 \times 20 \approx 83.021.$$

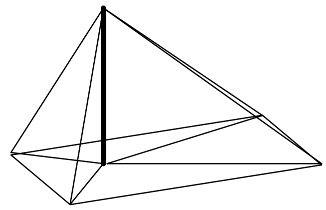
So the sum of the areas of the non-overlapping portions is

$$\pi\left(20^2 + 15^2\right) - 4 \times 83.021 \approx 1631 \text{ cm}^2.$$

M4 A radio ham places an aerial mast where it gives the best reception on the roof of his rectangular garage. He then fixes wire supports from the top of the mast to each corner of the roof. The lengths of two opposite supports are seven metres and four metres and the length of one of the others is 1 metre. Find the length of the remaining support.

Solutions





Let the length of the remaining wire support be x metres.

Let the height of the mast be h metres and the distances of the base of the mast from the sides of the roof be as shown in the diagram above.

Then

$$a^2 + c^2 + h^2 = 4^2$$

$$b^2 + c^2 + h^2 = 1^2$$

$$b^2 + d^2 + h^2 = 7^2$$

Hence

$$\left(a^2 + c^2 + h^2\right) + \left(b^2 + d^2 + h^2\right) - \left(b^2 + c^2 + h^2\right) = 16 + 49 - 1 = 64$$

The left-hand side reduces to $a^2 + d^2 + h^2$ but this is also the expression for x^2. So

$$x^2 = 64.$$

Hence $x = 8$ so the length of the remaining wire support is 8 metres.

M5 A large board has 1000 + signs and 999 − signs written on it. Any two symbols can be deleted provided they are replaced as follows:

- if the deleted symbols are the same, they are replaced by a +
- if the deleted symbols are different they are replaced by a −.

Repeat this process until there is only one symbol left. Which symbol is it and why?

Solutions

If two + signs are replaced by one +, the number of − signs remains unchanged.

If two − signs are replaced by one +, the number of − signs reduces by 2.

If a + sign and a − sign are replaced by one −, the number of − signs remains unchanged.

So in whatever order the signs are deleted, the number of − signs must always be odd since we started with 999, an odd number.

So when there is only one sign left, it must be a − sign to make the number of − signs odd.

(Whilst this solution is short, it is not easy to find!)