2015-2016 Middle Solutions Round 1

M1. On July 1st, Bill started to read a book recommended by friends, and, by reading the same number of pages each day of the month, managed to finish it on the 31st.

Another friend, Clare, also started reading the book at the same time. She read a quarter of that number of pages on the first of the month and, on each following day, one more page than on the previous day. She also finished on the 31st.

How many pages did the book contain?

Solution

Let x be the number of pages read each day by Bill.

$$31x = \frac{x}{4} + \left(\frac{x}{4} + 1\right) + \left(\frac{x}{4} + 2\right) + \left(\frac{x}{4} + 3\right) + \ \ldots \ + \left(\frac{x}{4} + 30\right)$$

$$\text{so} \quad 31x = 31 \times \frac{x}{4} + 1 + 2 + \ \ldots \ + 30$$

$$\text{so} \quad 31x = 31 \times \frac{x}{4} + 465$$

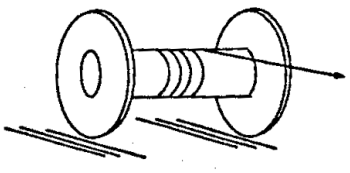
$$\text{so} \quad x = \frac{x}{4} + 15$$

$$\text{so} \quad \frac{3}{4}x = 15$$

$$\text{so} \quad x = 20.$$

Answer: the number of pages is $20 \times 31 = 620$.

M2.



An almost empty bobbin is pulled along a flat surface by a thread which is wrapped around it, as shown in the diagram. The diameter of the inner reel is 5 cm and that of each outer wheel is 10 cm. Assuming no slipping or sliding, how far has the bobbin moved when the end of the thread has moved 12 cm?

Solution

In one revolution the bobbin will move 10π cm on the flat surface.

This will unwind an additional 5π cm of thread.

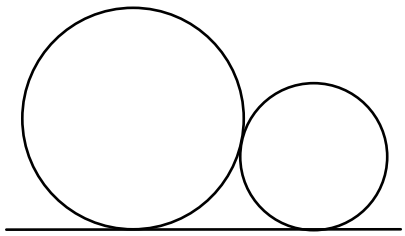
So the free end of the thread will move 15π cm.

Thus the bobbin moves 2/3 of the distance moved by the end of the thread.

i.e. When the end of the thread moves 12 cm the bobbin will move 8 cm.

(Try it!)

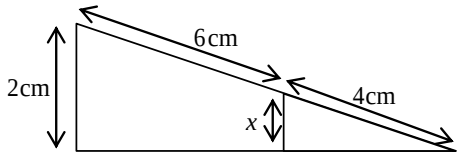
M3. Two cylinders, with diameters 12 cm and 8 cm, rest on a flat surface touching each other with their axes parallel. The diagram shows their cross-section.



What is the height of their point of contact above the surface?

Solution:

Join the centres to create this diagram.



Using similarity

$$\frac{x}{4} = \frac{2}{10} \Rightarrow x = 0.8.$$

The centre of the smaller cylinder is 4 cm above the horizontal surface, hence the height of their point of contact is 4.8 cm.

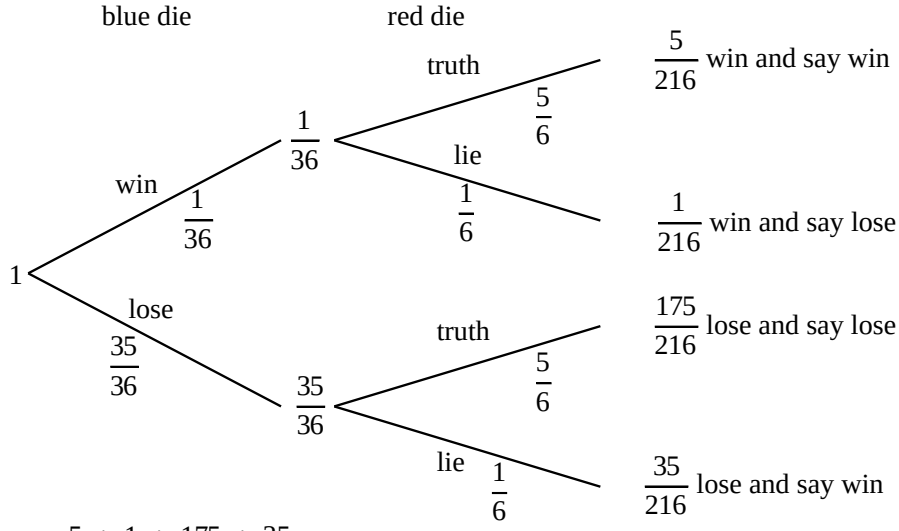
M4. I have two blue dice and one red die.
I use the blue dice to play a simple game: if I roll a double six, I win. Otherwise, I lose.
I also roll the red die. If I roll a one, I'll lie about whether I've won or lost the game; if I roll any other number, I'll tell the truth.

I roll all three dice.
I turn to you and say “I won!”.

What is the probability that I did in fact win the game?

Solution

P(win) = $\frac{1}{6} \times \frac{1}{6} = \frac{1}{36}$ (A)



$\frac{5 + 1 + 175 + 35}{216} = 1$

P(say win) = P(win and say win) + P(lose and say win)

$= \frac{5}{216} + \frac{35}{216} = \frac{40}{216}$ (B)

P(win given say win) = P(win and say win)/P(say win)

$= \frac{\frac{5}{216}}{\frac{40}{216}} = \frac{5}{40} = \frac{1}{8}$

So when I say I won the game, the probability that I did in fact win is $\frac{1}{8}$.

M5. A coach travels over a hilly route from town A in the highlands to town B by the coast. Going uphill it travels at 42 mph, going downhill it travels at 56 mph and on level ground it travels at 48 mph. It takes 2 hours and 20 minutes to travel from A to B and 2 hours and 40 minutes to travel back. Find the distance between A and B.

Solution

Let the distance from A to B be u miles up, d miles down and f miles on the flat.

Then

$$\frac{u}{42} + \frac{d}{56} + \frac{f}{48} = \frac{7}{3}$$

and on the return route where uphill and downhill exchange

$$\frac{d}{42} + \frac{u}{56} + \frac{f}{48} = \frac{8}{3}.$$

Adding:

$$(d + u)\left(\frac{1}{42} + \frac{1}{56}\right) + \frac{2f}{48} = 5$$

$$(d + u) \times \frac{2}{48} + \frac{2f}{48} = 5$$

$$d + u + f = \frac{5 \times 48}{2} = 120.$$

Hence the distance is 120 miles.

Now check that this is possible, since u could work out to be negative!

Subtracting:

$$\frac{d - u}{42} - \frac{d - u}{56} = \frac{1}{3}$$

$$(d - u)\left(\frac{1}{42} - \frac{1}{56}\right) = \frac{1}{3}$$

$$d - u = 56$$

If $u = 0$, then $d = 56$ and $f = 64$.

The hilly route suggests that $u > 0$, but that is certainly possible:

e.g. if $u = 28$, then $d = 84$ and $f = 8$

i.e. 28 miles uphill, 84 miles downhill and 8 miles on the flat.