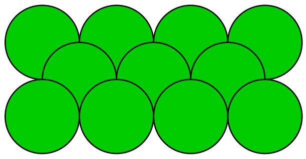
2014-2015 Middle Solutions Round 2

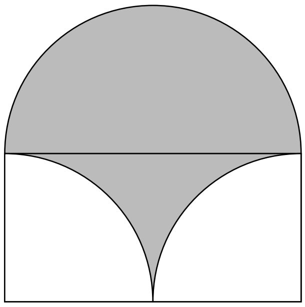
M1

The scales of a large fish are made up of arcs of circles with radius r cm. Each row of scales overlaps the row below. The scales within a row just touch, with their centres on a straight line. The next row of scales, which overlaps the previous row, is r cm above the previous row, with the centres of the scales above the points where the scales in the previous row touch.

What is the visible area of a single scale?

Solution





The area of the semicircle is $\frac{1}{2}\pi r^2$.

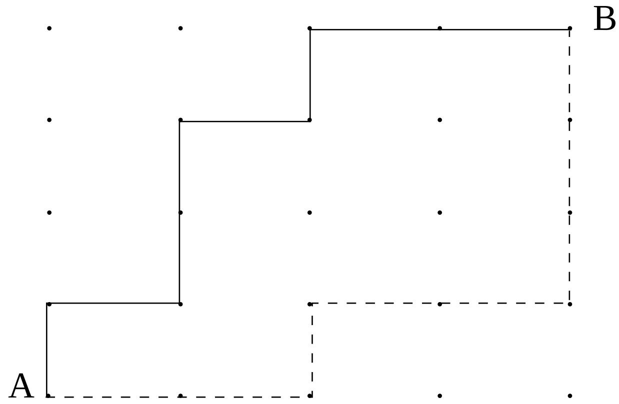
The rectangle is $2 \times 2r$ so its area is $2r^2$.

So the visible area of the top rectangle is $2r^2 - \frac{1}{2}\pi r^2$.

Hence the shaded area is

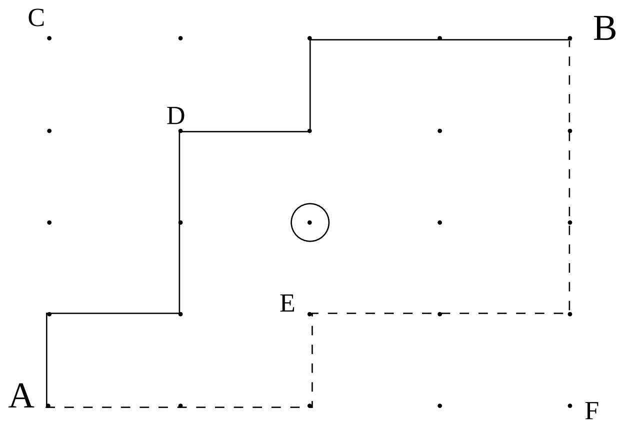
$$\frac{1}{2}\pi r^2 + \left(2r^2 - \frac{1}{2}\pi r^2\right) = 2r^2.$$

M2



Paths from A to B can **only** proceed upwards or to the right: two example paths from A to B are shown. How many such paths are there from A to B that do not go through the centre dot?

Solution



There is only 1 path from A to B via C (or F).

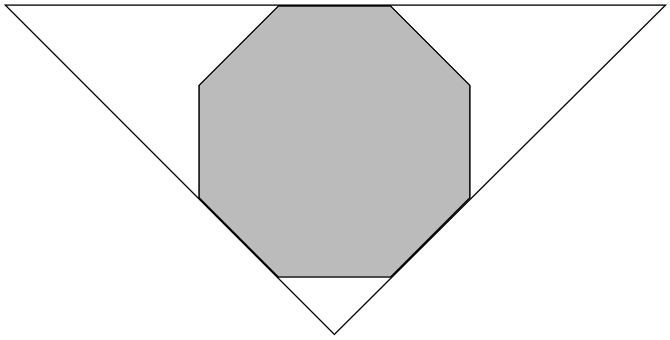
There are 4 paths from A to D (one with its right moving section at each of the 4 levels). Each of these can be combined with one of the 4 paths from D to B, making $4 \times 4 = 16$ possible paths via D (or E).

Thus there are 1 + 16 + 16 + 1 = 34 paths in all not passing through the centre point.

M3

Three sides of a regular polygon with 8 sides are chosen at random. Find the probability that, when these sides are extended, they form a triangle containing the polygon.

Solution



The only arrangement of edges which forms a triangle enclosing the octagon is as above. (If two adjacent or two opposite edges are chosen, there is no third edge which completes a triangle.)

There are $8 \times 7 \times 6/3! = 56$ ways of selecting sets of three edges.

The long edge of the triangle could be along any one of the 8 edges of the octagon, and so there are 8 ways of selecting a set of three edges which enclose the octagon.

So the probability is $\frac{8}{56} = \frac{1}{7}$.

M4

A fruit drink manufacturer has a mixture of 100 litres containing w% of pure orange juice. By adding x litres of a mixture containing y% of pure orange juice he wishes to produce a mixture containing z% of pure orange juice. Find the value of x in terms of w, y and z.

Solution

	litres of mixture	% pure orange juice	litres of pure orange juice
start	100	w	w
amount added	x	y	$\frac{xy}{100}$
total	$100 + x$	z	$w + \frac{xy}{100}$

The number of litres of pure orange juice in the final mixture is also

$$\frac{(100 + x)z}{100}$$

Hence

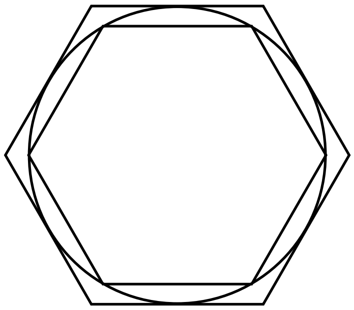
$$\frac{(100 + x)z}{100} = w + \frac{xy}{100}$$

$$(100 + x)z = 100w + xy$$

$$(z - y)x = 100w - 100z = 100(w - z)$$

$$x = \frac{100(w - z)}{z - y}$$

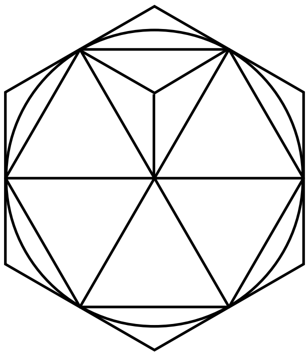
M5



A regular hexagon circumscribes a circle which circumscribes another regular hexagon. The inner hexagon has an area of 3 square units. What is the area of the outer hexagon?

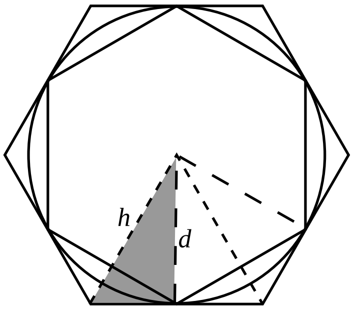
Solution

First, rotate the outer hexagon so that the vertices of the inner hexagon touch the outer hexagon. The radial line of the circle to a vertex of the inner hexagon is then at right angles to the edge of the outer hexagon. If one of the equilateral triangles of the inner hexagon is split into three isosceles triangles, as shown, the angle of each of these isosceles triangles at a vertex of the hexagon is 30 degrees. Hence the angle at that vertex of the triangle outside the inner hexagon is also 30 degrees. So the area of the triangle outside the inner hexagon is equal to the area of one of these isosceles triangles. So the area of the outer hexagon is 4/3 times the area of the inner hexagon i.e. is 4 square units.



(This can, of course, also be done without the initial rotation).

Alternative



Each hexagon can be split into six equilateral triangles. Let the sides be h and d as indicated.

The area of each of the larger triangles is $\frac{1}{2}h^2 \sin 60°$ and the area of each of the smaller triangles is $\frac{1}{2}d^2 \sin 60°$. Thus

$$\frac{\text{Area of larger hexagon}}{\text{Area of smaller hexagon}} = \frac{6 \times \left(\frac{1}{2}h^2 \sin 60°\right)}{6 \times \left(\frac{1}{2}d^2 \sin 60°\right)} = \frac{h^2}{d^2}.$$

But, from the shaded right-angled triangle

$$\frac{d}{h} = \sin 60° = \frac{\sqrt{3}}{2} \Rightarrow \frac{h}{d} = \frac{2}{\sqrt{3}} \Rightarrow \frac{h^2}{d^2} = \frac{4}{3}.$$

Thus

$$\text{Area of larger hexagon} = \frac{4}{3} \times \text{Area of smaller hexagon} = \frac{4}{3} \times 3 = 4.$$