2014-2015 Middle Round 1 solutions

M1

A shop sells sweets in bags of 7 and 20. What is the largest number of sweets that cannot be purchased exactly? **Justify your answer.**

Solution

7, 14, 21, 28, … can all be purchased in bags of 7 sweets i.e. all amounts that are multiples of 7.

If I start with a bag of 20 sweets, I can purchase 20, 27, 34, … i.e. amounts one less than a multiple of 7.

If I start with 2 bags of 20 sweets, I can purchase 40, 47, 54, … i.e. amounts two less than a multiple of 7.

…

If I start with 6 bags of 20 sweets, I can purchase 120, 127, 134, … i.e. amounts six less than a multiple of 7.

119 is a multiple of 7 so I can purchase all amounts more than 120.

And the largest amount I cannot purchase is 120 - 7 = 113 sweets.

M2

If a year had only 364 days then we could use the same calendar every year. But actually most years have 365 days, and leap years have 366 days. For the relevant years, a leap year occurs when the year is divisible by 4.

I was just about to throw away my calendar for 2014 when I wondered when I would first be able to reuse it. In which year will that be?

Finding I had a copy of each calendar from the year 2000, I wondered further about this problem. What would be the greatest amount of time I would have to keep any calendar until it could be re-used? **Justify your answer.**

Solution

year	1st March day	
2014	n	Sunday
2015	$n + 1$	Monday
2016	$n + 3$	Wednesday
2017	$n + 4$	Thursday
2018	$n + 5$	Friday
2019	$n + 6$	Saturday
2020	$n + 8 \equiv n + 1$	Monday
2021	$n + 2$	Tuesday
2022	$n + 3$	Wednesday
2023	$n + 4$	Thursday
2024	$n + 6$	Saturday
2025	$n + 7 = n$	Sunday

So 2025 has the 1st March on the same day as 2014. Neither year is a leap year, and so the 2014 calendar can be used again in 2025. 2014 is a leap year + 2, and any leap year + 2 calendar can be reused after 11 years.

Having shown from above what happens for non leap years, consider what happens when the calendar contains 366 days.

Start 2016 (could be any leap year but using information from before)

2016	$n + 3$	
2020	$n + 8$	$(n + 1)$
2024	$n + 13$	$(n + 6)$
2028	$n + 18$	$(n + 4)$
2032	$n + 23$	$(n + 2)$
2036	$n + 28$	(n)
2040	$n + 33$	$(n + 5)$
2044	$n + 38$	$(n + 3)$

We also need to consider a leap year + 1 and a leap year + 3.

year	day
2001	n
2002	$n + 1$
2003	$n + 2$
2004	$n + 4$
2005	$n + 5$
2006	$n + 6$
2007	$n + 7 (\equiv n)$

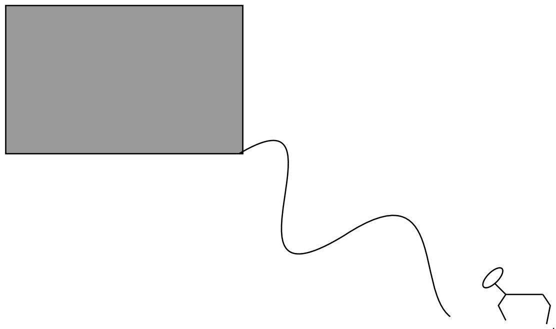
So a leap year + 1 calendar can be reused after 6 years.

year	day
2003	n
2004	$n + 2$
2005	$n + 3$
2006	$n + 4$
2007	$n + 5$
2008	$n + 7 \equiv n$
2009	$n + 1$
2010	$n + 2$
2011	$n + 3$
2012	$n + 5$
2013	$n + 6$
2014	$n + 7 \equiv n$

So a leap year + 3 calendar can be reused after 11 years.

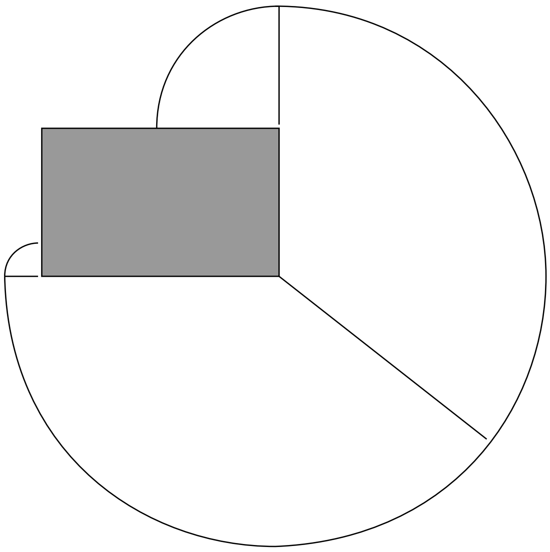
Hence the longest time before a calendar can be reused is 28 years from when it first came into use.

M3



A goat is tied to the corner of a rectangular shed as shown. The shed is 9 metres long and 7 metres wide and the rope is 10 metres long. The shed is surrounded by grass. Find the area of grass that the goat can graze on.

Solution

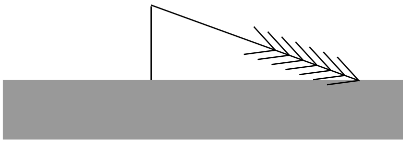


The goat can eat a three-quarter circle of radius 10 m, i.e. area $\frac{3}{4} \times \pi \times 10^2\ \text{m}^2 = 75\pi\ \text{m}^2$.
And a quarter circle of radius 3 m, i.e. area $\frac{1}{4} \times \pi \times 3^2\ \text{m}^2 = 2\frac{1}{4}\pi\ \text{m}^2$.
And a quarter circle of radius 1 m, i.e. area $\frac{1}{4} \times \pi \times 1^2\ \text{m}^2 = \frac{1}{4}\pi\ \text{m}^2$

Thus the total area the goat can eat is $77\frac{1}{2}\pi\ \text{m}^2$ ($\approx$ 243.5 m^2).

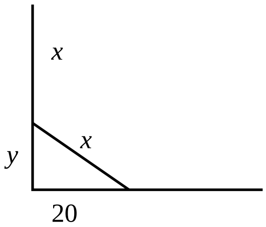
M4

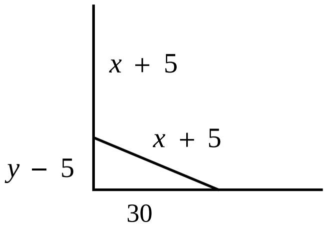
During a hurricane, a telegraph pole was broken in such a way that the top struck the level ground at a distance of 20 feet from the base of the pole. It was replaced by an identical pole which was broken by another gale at a point 5 feet lower down and the top struck the ground a distance of 30 feet from the base. What was the original height of the poles?



Solution

Let x be the length of the part snapped off the first pole and y the height of the break above the ground.





From original: $x^2 - y^2 = 20^2 = 400$

From second:

$$(x + 5)^2 - (y - 5)^2 = 30^2 = 900$$

$$x^2 + 10x + 25 - \left(y^2 - 10y + 25\right) = 900$$

$$\left(x^2 - y^2\right) + 10(x + y) = 900$$

$$400 + 10(x + y) = 900$$

$$10(x + y) = 500$$

$$x + y = 50$$

The poles have height 50 feet.

M5

Two ferry boats set out at the same time from opposite banks of a loch. One boat is faster than the other and they pass each other at a point 650 metres from the nearer bank. After arriving at their destinations, each boat remains for 15 minutes to change passengers and then sets out on the return journey. This time, they meet at a point 350 metres from the other bank. How wide was the loch?

Solution

Since each boat spends the same time at rest, we can ignore that time and assume that they simply turn round and sail back immediately.

Let the width of the river be W metres and let the speeds of the boats be u and v in metres per second respectively, where we assume $u \leqslant v$.

Then, on the first crossing, the time taken to meet is

$$\frac{650}{u} = \frac{W - 650}{v} \Rightarrow \frac{u}{v} = \frac{650}{W - 650}$$

Similarly, on the second crossing time taken to meet is

$$\frac{W + 350}{u} = \frac{W + W - 350}{v} \Rightarrow \frac{u}{v} = \frac{W + 350}{2W - 350}$$

Hence

$$\frac{650}{W - 650} = \frac{W + 350}{2W - 350}$$

$$650(2W - 350) = (W + 350)(W - 650)$$

$$1300W - 650 \times 350 = W^2 - 300W - 350 \times 650$$

$$1600W = W^2$$

which, as $W \neq 0$, gives $W = 1600$ so the width of the loch is 1600 metres