# 2013-2014 Middle Problems 2

## M1.

A block of four postage stamps, with perforations along the joins so that they can be easily separated, have values in pence as shown:

| | |
|---|---|
| 1 | 2 |
| 4 | 3 |

Show that it is possible to make every postage value from 1p to 10p using either a single stamp or a number of stamps joined along lines of perforations.

Using a different set of stamp values in the block of four, it is possible to make every postage value from 1p to a higher limit than 10p. Construct an arrangement of stamp values which reaches the highest possible limit.

Are there any other solutions which give this limit? Explain.

*Solution*

1p, 2p, 3p and 4p are single stamps.

5p: 1p and 4p
6p: 1p and 2p and 3p
7p: 4p and 3p
8p: 1p and 4p and 3p
9p: 2p and 3p and 4p
10p: 1p and 2p and 3p and 4p

To find the maximum possible total, first count the number of combinations of stamps available:

| 4 single stamps | 4 joined pairs of stamps | 4 joined threes | all 4 stamps |
|---|---|---|---|
| 4 | 4 | 4 | 1 |

so there is a total of 13 combinations, so highest possible total value is 13p, with each value being obtained in exactly one way.

But can we arrange the stamp values to achieve this?

There must be a 1p stamp, otherwise we could not have this value.

If there were another 1p stamp, then there would be 2 possible ways to get 1p and another value would be missed. So there is only one 1p stamp.

The next lowest value stamp must then be 2p.

Case 1: Put the 2p stamp next to the 1p stamp:

| | |
|---|---|
| 1 | 2 |
| | |

This gives a total of 3p from these 2 joined stamps, so the next lowest value stamp must be 4p, and the final stamp has value $13 - 1 - 2 - 4 = 6$ pence. There can be only one way to get 6p value, and so the 4p stamp cannot be next to the 2p stamp, making the final arrangement:

| | |
|---|---|
| 1 | 2 |
| 4 | 6 |

Does this arrangement work? 5p is obtained from 1p and 4p joined, and 7p up to 12p are what is left when 1p to 6p are taken from 13p. Finally 13p is obtained from all four stamps. So yes.

Case 2: Put the 2p stamp diagonally opposite the 1p stamp. The next stamp value must then be 3p, and the final stamp value $13 - 1 - 2 - 3 = 7$ pence. The positions are then

| | |
|---|---|
| 1 | 3 |
| 7 | 2 |

Finally check that all values can actually be obtained from joined stamps:

1p, 2p, 3p, 7p single stamps, 4p: 1p and 3p
5p: 2p and 3p 13p: all four stamps

and the remaining values are left behind when others are detached.

There are no further different possible positions for the 2p stamp (it must be either next to or diagonally opposite the 1p stamp) and so we have found the only two sets of stamp values possible.

## M2.

A brother and sister, Peter and Fiona, are always thinking about numbers.

On his birthday Peter said “My age is a square number.”

His older sister Fiona said “That's right, but the sum of our ages and the difference of our ages also give squared numbers.”

Peter replied “In three years time, both our ages will be prime numbers.”

Fiona replied “Three years ago, both our ages were also prime numbers.”

What are the ages of Peter and Fiona now?

*Solution*

(We assume throughout that Peter and Fiona are not aged well over a hundred.)

Peter's current age is a square number but in three years, it will be a prime so his current age must be even.

| | | | | | |
|---|---|---|---|---|---|
| Possible ages | 4 | 16 | 36 | 64 | 100 |
| In 3 years time | 7 | 23 | 39 | 67 | 103 |
| 3 years before | 1 | 13 | 33 | 61 | 97 |

So Peter is either 16 or 64 or 100.

In three year's time, Fiona's age will be prime and therefore odd so at present it is even. Let it be $x$, and if Peter is 16 then $x - 16$ and $x + 16$ are both square numbers. So we need two square numbers which differ by 32.

| | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|
| Square numbers | 1 | **4** | 9 | 16 | 25 | 36 | **49** | 64 | 81 | 100 |
| + 32 | 33 | **36** | 41 | 48 | 57 | 68 | **81** | 96 | 113 | 132 |

So from this, if Peter is 16 then Fiona is 20 or 65 (impossible because odd).

But if Peter is 64 then $x - 64$ and $x + 64$ are both square numbers. So we need two square numbers which differ by 128.

| | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| Square numbers | 1 | 4 | 9 | **16** | 25 | 36 | 49 | 64 | 81 |
| + 128 | 129 | 133 | 137 | **144** | 163 | 164 | 177 | 192 | 209 |

So from this, if Peter is 64 then Fiona is 80. But three years ago, Fiona would have been 77 which is not prime. So Peter is not 64.

But if Peter is 100 then $x - 100$ and $x + 100$ are both square numbers. So we need two square numbers which differ by 200.

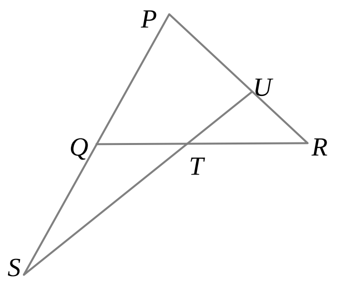
| | | | |
|---|---|---|---|
| Square numbers | 1 | 4 | 9 |
| + 200 | 201 | 204 | 209 |

So from this, if Peter were 100 there are no pairs of squares so Peter is not 100.

## M3.

$PQR$ is any triangle. The side $PQ$ is extended to $S$ where $PQ = QS$. The point $U$ divides the side $PR$ in the ratio $3 : 2$. The point $T$ is where the lines $QR$ and $SU$ cross.

Find the ratio $\dfrac{QT}{QR}$.




*Solution*

Construct $SV$ parallel to $QR$ to meet $PR$ produced at $V$. (This is the tricky bit!)

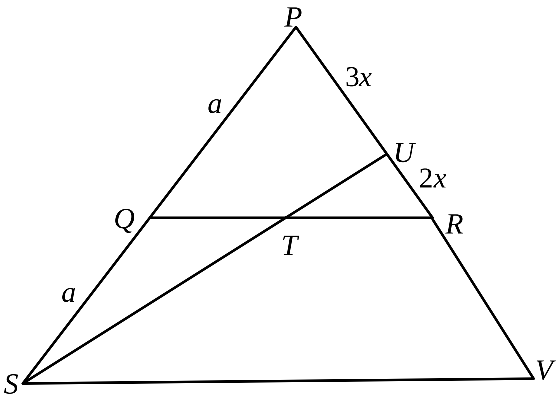
From similar triangles $PSV$ and $PQR$, because $PS = 2PQ$, $PV = 2PR$. So $RV = 5x$.

Also $SV = 2QR$.

From similar triangles $UTR$ and $USV$, $TR = \frac{2}{7}SV = \frac{4}{7}QR$.

Therefore $QT = QR - TR = \frac{3}{7}QR$.

i.e. $\dfrac{QT}{QR} = \dfrac{3}{7}$.




*Alternative*:

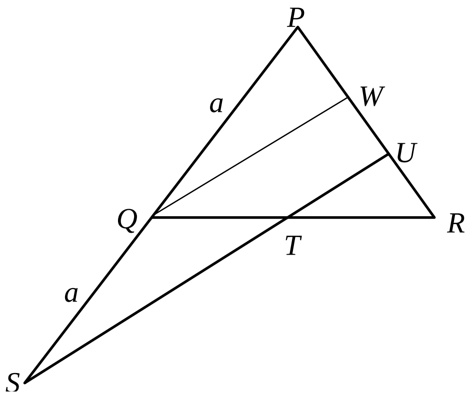
Draw $QW$ parallel to $SU$ to meet $PR$ at $W$.

Then $\dfrac{QT}{TR} = \dfrac{WU}{UR}$ from triangles $QRW$, $TRU$.

Also $\dfrac{WU}{PU} = \dfrac{QS}{PS} = \dfrac{1}{2}$ from triangles $QWP$, $SUP$.

Hence $WU = \frac{1}{2}PU$.

So $\dfrac{QT}{TR} = \dfrac{PU}{2\,UR} = \dfrac{3}{4}$ and therefore $\dfrac{QT}{QR} = \dfrac{3}{7}$.

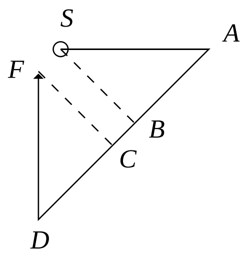


## M4.

Rebecca celebrated her graduation by going for a hot air balloon ride. At first the wind blew the balloon $\frac{1}{2}$ a mile due east. Then the balloon was blown $\frac{3}{4}$ of a mile southwest. Finally, it was blown $\frac{1}{2}$ a mile north and then landed. How many miles did the balloon land from the point where it was launched? Express your answer as a decimal to three places.

**Do not use a scale drawing.**

*Solution*




Draw lines through the start point and the finish point which are perpendicular to the southwest flight.

Let $x$ be the length of $SB$.

Since the angle between east and southwest is 45°, triangle $SAB$ is isosceles so $AB = x$.

Then $x^2 + x^2 = \frac{1}{2}^2$ so $x^2 = \frac{1}{8}$ so $AB = \sqrt{\frac{1}{8}} = \frac{1}{2\sqrt{2}}$

The same argument can be applied to triangle $CDF$ to give $CD = \frac{1}{2\sqrt{2}}$.

Since $BCFS$ is a rectangle, $SF = BC$.

So the required distance is $AD - AB - CD = \frac{3}{4} - 2x = \frac{3}{4} - \frac{1}{\sqrt{2}} \approx 0{\cdot}043$.

## M5.

Dots are arranged in a rectangular grid with 4 rows and *n* columns. Consider different ways of colouring the dots, in which each dot either red or blue. A colouring is 'good' if no four dots of the same colour form a rectangle with horizontal and vertical sides.

Find the maximum value of *n* for which there is a good colouring.

*Solution*

First note that a grid with a repeated column has four dots of the same colour at the corners of a rectangle.

Consider a grid with only 3 rows. Then the only possible columns are

r r b b b r r b
r b r b r b r b
r b b r r r b b

The middle 6 columns form the largest good colouring.

Adding an extra row can never turn a bad colouring good, and so a good grid with 4 rows can have at most 6 columns.

r b b b r r
b r b r b r
b b r r r b
r r r b b b

This example with 4 rows shows that a good grid with 6 columns is possible.

Hence 6 is the maximum value of *n* for which there is a good colouring.