2013-2014 Middle Solutions Round 1

M1

In a recent election, six candidates stood and a total of 51880 votes were cast. The winning candidate beat the others by 1336, 7085, 15333, 15654 and 17102 votes respectively. Candidates lose their deposit if they fail to get more than 5% of the total number of votes cast. How many candidates lost their deposits?

Solution

Let v be the number of votes obtained by the winner. So the total number of votes cast was

$$v + (v - 1336) + (v - 7085) + (v - 15333) + (v - 15654) + (v - 17102) = 6v - 56510.$$

The total number of votes cast was 51880. Hence

$$6v - 56510 = 51880$$

$$\text{i.e.} \quad 6v = 56510 + 51880 = 108\,390$$

$$\text{i.e.} \quad v = 18065.$$

So the six candidates got the following numbers of votes:

18065   18065 – 1336 = 16729   18065 – 7085 = 10980   18065 – 15333 = 2732
18065 – 15654 = 2411   18065 − 17102 = 963.

As 5% of the total is 2594, two candidates each lost their deposit.

M2

In a tennis tournament, each match is played between two players, and the winner proceeds to the next round whereas the loser is eliminated. There are no draws. If necessary, in the first round only, a number of players do not participate.

(a) A particular tournament starts with 256 players and proceeds until there is one overall winner. How many matches are played in this tournament?

(b) If the tournament starts with 296 players and proceeds until there is one overall winner. How many matches are played in this tournament?

Solution

(a) Starting with 256 players, if all play in the 128 first round matches, the number of matches in successive rounds will be 64, 32, 16, 8, 4, 2, 1. 1 mark.

Number of matches played = 128 + 64 + 32 + 16 + 8 + 4 + 2 + 1 = 255 matches 1 mark

(b) In this case there are 40 more players than in (a) so there must be 40 more matches. Thus there are 295 matches.

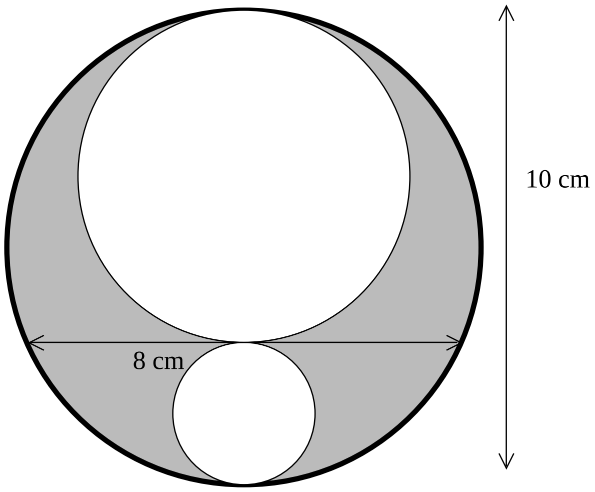
Alternative Solution

(a) Each match eliminates one player.
So to eliminate 255 players 255 matches are required.

(b) 295 matches

M3

A jeweller has been asked to make a pendant in the shape of the shaded area shown. The height of the pendant is 10 cm and the distance across at the point where the two smaller circles touch is 8 cm.



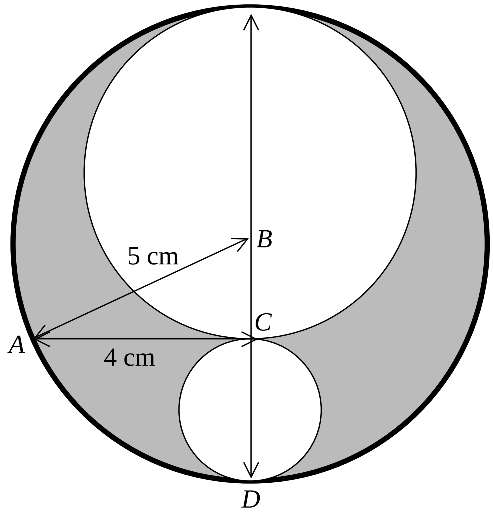
Find the area of the pendant.

Solution

Radius of large circle = 5 cm so $BD = BA = 5$.

Half of the length of the chord = 4 cm so $AC = 4$. Applying Pythagoras' Theorem to ΔABC gives $BC = 3$, $CD = 2$ and so the diameter of the middle circle is 8 cm and its radius is 4 cm.

Radius of smallest circle = 1 cm.



Shaded area

$$= \pi\left(5^2 - 4^2 - 1^2\right) = 8\pi \text{ cm}^2$$

$$\approx 25.12 \text{ cm}^2$$

M4

On a tiny remote island where the death sentence still exists a man can be granted mercy after receiving the death sentence in the following way:

- he is given 18 white balls and 6 black balls.
- he must divide them between three boxes with at least one ball in each box.
- he is then blindfolded and must choose a box at random and then a single ball from within this box.

He receives mercy only if the chosen ball is white.

Find the probability that he receives mercy when he distributes the balls in the most favourable manner.

Solution

If he puts all the black balls in one box and the whites in the other two the probability of reprieve is $\frac{2}{3}$.

If he distributes the balls evenly (6w and 2b in each box) the probability of reprieve is $\frac{3}{4}$ which is better.

For a better chance, he should make sure that two of the boxes contain only white balls, one in each.

So he should put all the black balls in the third box and then he should put all but two white balls in the third box as well to improve his chances there.

So he should be certain of reprieve with 2 boxes (1w in each) and put the remaining balls in the third (16w and 6b). Then the probability is

$$\frac{1}{3} + \frac{1}{3} + \frac{1}{3} \times \frac{16}{22} = \frac{10}{11}.$$

M5

Three types of item, A, B and C, are for sale. Items of type A sell at 8 for £1. Items of type B sell for £1 each. Items of type C sell for £10 each. A selection of 100 items which includes at least one of each type costs £100. How many items of type B are there in the selection?

Solution

Let a be the number of items of type A in the selection.

Let b be the number of items of type B in the selection.

Let c be the number of items of type C in the selection.

$$\frac{a}{8} + b + 10c = 100$$

$$a + b + c = 100$$

So

$$b = 100 - a - c$$

$$\frac{a}{8} + 100 - a - c + 10c = 100$$

$$-\frac{7}{8}a + 9c = 0$$

$$7a = 72c$$

Either $a = c = 0$, which would mean that there were item types missing from the selection, and hence is not possible.

Or $a = 72$ and $c = 7$ (larger multiples of these are not possible as there are only 100 items in all.)

Hence $b = 100 - 72 - 7 = 21$

i.e. there are 21 items of type B.